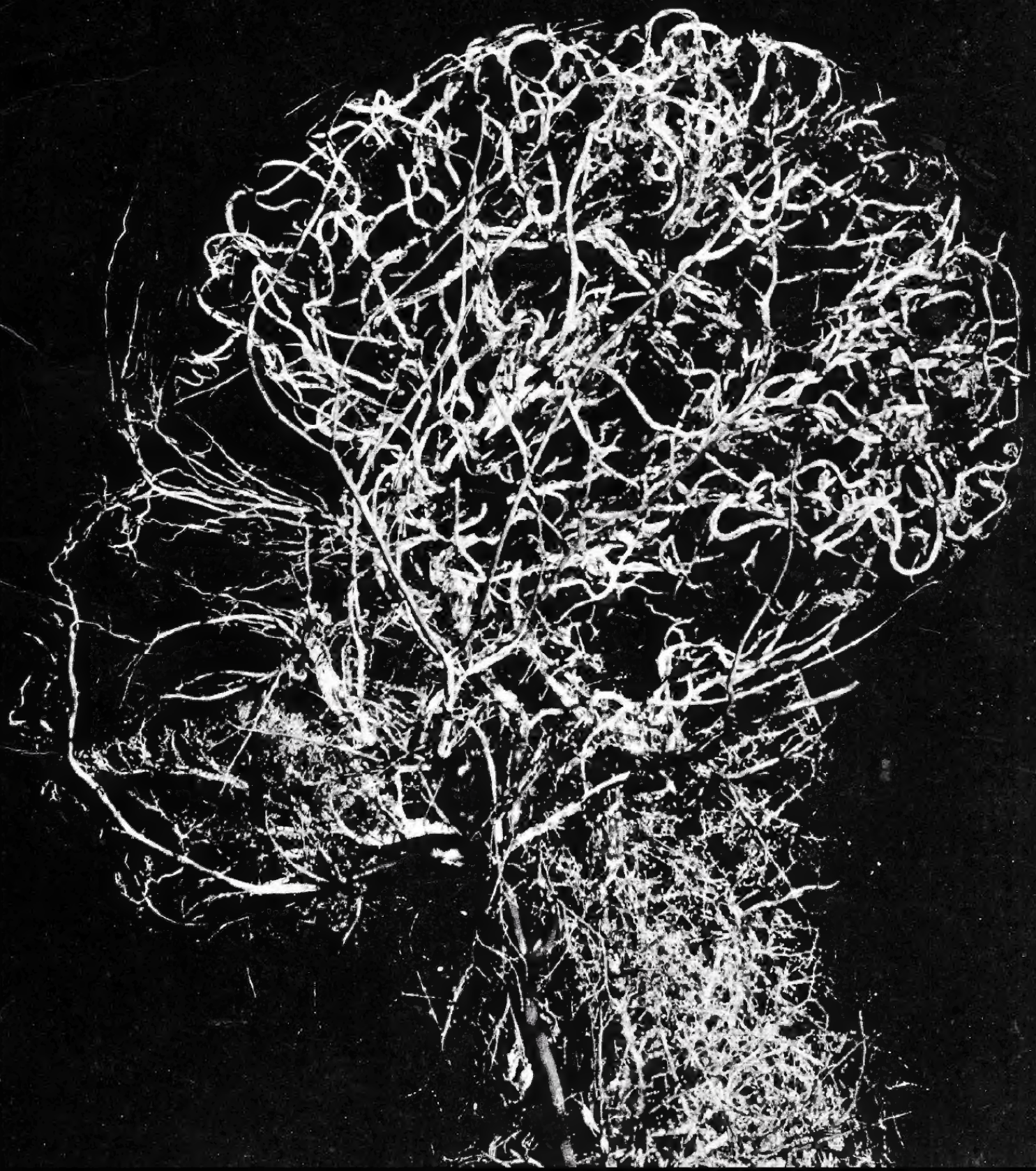
M321 15 and 16

THE OPEN UNIVERSITY

Mathematics: A Third Level Course
Partial Differential Equations of Applied Mathematics Units 15 and 16

# Parabolic Equations
# Blood Flow



THE OPEN UNIVERSITY

*Mathematics: A Third Level Course*

*Partial Differential Equations of Applied Mathematics Units 15 and 16*

# PARABOLIC EQUATIONS
# BLOOD FLOW

*Prepared by the Course Team*

The Open University Press

# Unit 15 Finite Differences IV: Parabolic Equations

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## Set Books

G. D. Smith, *Numerical Solution of Partial Differential Equations* (Oxford, 1971).

H. F. Weinberger, *A First Course in Partial Differential Equations* (Xerox, 1965).

It is essential to have these books; the course is based on them and will not make sense without them. They are referred to in the text as *S* and *W* respectively.

*Unit 15* includes a short reading passage from *S*, but is not otherwise based on either set book.

## Conventions

Before working through this text make sure you have read *A Guide to the Course: Partial Differential Equations of Applied Mathematics*. References to Open University courses in mathematics take the form:

*Unit M100 13, Integration II* for the Mathematics Foundation Course,

*Unit M201 23, The Wave Equation* for the Linear Mathematics Course.

# 15.0 INTRODUCTION

In the three previous numerical units we have looked at some simple partial differential equations and developed finite-difference schemes to solve them. We have been interested primarily in the conditions under which our numerical methods can be used with confidence. This has led us to discuss stability and convergence in the solution of initial value problems and convergence of iterative methods in various contexts.

In this unit we shall look at some of the problems we have neglected, dealing exclusively with parabolic equations.

We begin the unit by attempting to improve the accuracy of finite-difference schemes applied to parabolic equations in one space dimension. This investigation leads us naturally into a discussion of *three-level schemes*. It so happens that three-level schemes are useful in overcoming some of the difficulties associated with the numerical solution of nonlinear partial differential equations and so we devote much of the discussion to the treatment of a simple class of such equations. This section will illustrate the usefulness of finite-difference schemes in an area where it is extremely difficult to obtain analytical solutions. We shall also devote a short section to equations in two dimensions.

The final section is a discussion of what happens when the data involve discontinuities. In previous work, we have been concerned mainly with problems in which the differential equations have no discontinuities in their coefficients, and in which the subsidiary conditions are differentiable. There are, of course, problems in which these conditions are not satisfied and we shall discuss briefly the modifications which must be made to our finite-difference schemes to deal with these situations. As one might expect, there is no universal panacea for such problems and we have to rely heavily on our knowledge of the analytical properties of partial differential equations to give us some insight into possible lines of approach in individual cases. Indeed, it is always important to consider analytical and numerical techniques in conjunction with each other, for they can interact to suggest suitable techniques of solution.

## 15.1 PARABOLIC EQUATIONS IN ONE SPACE DIMENSION

### 15.1.1 Three-Level Schemes

In our earlier numerical work on parabolic equations, we concentrated on two-level schemes; that is, schemes which involve combinations of pivotal values on two time levels. We first introduced explicit schemes, which are easy to compute but give rise to problems of instability. Stable explicit methods require small step sizes in the $t$-direction and hence involve much computational effort. Implicit methods, on the other hand, give more involved computations at each step, but they allow us to use larger values of the step size because of good stability properties. One of the most important implicit methods is the Crank–Nicolson method which, as well as being unconditionally stable, has a small local truncation error. It should be clear that to be able to reduce the size of local truncation error of a scheme whilst retaining stability is of great value in the numerical solution of partial differential equations. This is because we can either obtain better accuracy in a given number of steps or get similar results with less effort by taking larger steps.

In the case of the simple *diffusion equation*,

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},$$

we saw in *Unit 5, Initial Value Problems* that replacement of the time derivative using the forward-difference formula

$$\frac{\partial U_{i,j}}{\partial t} \simeq \frac{u_{i,j+1} - u_{i,j}}{k},$$

where $k$ is the mesh spacing in the $t$-direction, leads to the simple explicit scheme

$$u_{i,j+1} = u_{i,j} + r(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) \tag{1}$$

where $r = k/h^2$. If we were to replace the time derivative using the more accurate central-difference formula

$$\frac{\partial U_{i,j}}{\partial t} \simeq \frac{u_{i,j+1} - u_{i,j-1}}{2k},$$

then we would obtain the following explicit scheme:

$$u_{i,j+1} = 2r(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + u_{i,j-1}. \tag{2}$$

It is immediately clear that this is a **three-level scheme** since it involves terms along the $(j-1)$th, $j$th and $(j+1)$th time levels. Our quest for higher accuracy has led us naturally to a three-level scheme and we must now see whether we have in fact produced a scheme with a smaller local truncation error. This is the subject of the following SAQ.

*SAQ 1*

Show that the simple three-level explicit scheme given by Equation (2) is locally more accurate than the two-level explicit scheme given by Equation (1) when both are applied to the diffusion equation.

(Solution on p. 24.)

Although we have produced a more accurate explicit scheme, it is of little use unless it turns out to be stable for some values of the mesh ratio $r$, and it is this aspect of three-level schemes which we shall now discuss. Rather than look at the explicit scheme on its own we shall include an analysis which works also for implicit three-level schemes.

By analogy with the work in Section 8.4.1 of *Unit 8, Stability* you can see that a general three-level scheme can be written in matrix form as

$$A\mathbf{u}_{j+1} = B\mathbf{u}_j + C\mathbf{u}_{j-1} \tag{3}$$

or, provided det $A \neq 0$, as

$$\mathbf{u}_{j+1} = A^{-1}B\mathbf{u}_j + A^{-1}C\mathbf{u}_{j-1}.$$

(For simplicity in the analysis we are considering the case of homogeneous boundary conditions, that is, $U(0, t) = U(1, t) = 0$.) By putting $\mathbf{v}_{j+1} = \mathbf{u}_j$ we can express this three-level scheme as a pair of two-level schemes, which can be written in partitioned matrix form as

$$\begin{bmatrix} \mathbf{u}_{j+1} \\ \mathbf{v}_{j+1} \end{bmatrix} = \begin{bmatrix} A^{-1}B & A^{-1}C \\ I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \tag{4}$$

where $I$ is the unit matrix of the same order as the matrices $A$, $B$ and $C$. We call the matrix $P$, given by

$$P = \begin{bmatrix} A^{-1}B & A^{-1}C \\ I & 0 \end{bmatrix},$$

the **amplification matrix**, and we know from Section 8.4.1 of *Unit 8*, which dealt with the matrix method for examining stability, that the scheme (4) is stable provided the *spectral radius* of the amplification matrix $P$ is not greater than unity. Therefore, we now need to find the eigenvalues of $P$. If $\lambda$ is an eigenvalue of $P$ then we may write

$$\begin{bmatrix} A^{-1}B & A^{-1}C \\ I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$

where $(\mathbf{u}, \mathbf{v})$ is an eigenvector. Expanding the equation, we obtain

$$A^{-1}B\mathbf{u} + A^{-1}C\mathbf{v} = \lambda\mathbf{u},$$

$$\mathbf{u} = \lambda\mathbf{v}.$$

Eliminating $\mathbf{u}$ between the equations, we are left with

$$(\lambda^2 I - \lambda A^{-1}B - A^{-1}C)\mathbf{v} = \mathbf{0},$$

and the eigenvalues of $P$ satisfy

$$\det(\lambda^2 I - \lambda A^{-1}B - A^{-1}C) = 0,$$

whence

$$\det(\lambda^2 A - \lambda B - C) = 0.$$

It is this equation which we find the most useful for determining the eigenvalues of the matrix $P$.

**Example**

Determine the eigenvalues of the amplification matrix of the three-level explicit scheme given by Equation (2) for the diffusion equation with homogeneous boundary conditions. Hence investigate the stability of this scheme.

*Solution*

Writing the scheme in the form of Equation (3) we see that $A = C = I$ and $B$ is given by

$$B = 2r \begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}.$$

The eigenvalues of the amplification matrix are given by

$$\det(\lambda^2 I - \lambda B - I) = 0$$

which, on dividing by $\lambda$ (assuming $\lambda \neq 0$), gives

$$\det\left(B - \frac{\lambda^2 - 1}{\lambda} I\right) = 0.$$

Therefore, if $\mu$ is an eigenvalue of $B$, then $\lambda$ satisfies

$$\frac{\lambda^2 - 1}{\lambda} = \mu.$$

The eigenvalues of $B$ are given by

$$\mu_s = -8r \sin^2\left(\frac{s\pi}{2M}\right) \qquad s = 1, 2, \ldots, M - 1$$

where $B$ has $M - 1$ rows and columns (see Appendix to *Unit 8*), and so

$$\lambda_s^2 + \lambda_s 8r \sin^2\left(\frac{s\pi}{2M}\right) - 1 = 0.$$

Hence,

$$\lambda_s^{\pm} = -4r \sin^2\left(\frac{s\pi}{2M}\right) \pm \sqrt{1 + 16r^2 \sin^4\left(\frac{s\pi}{2M}\right)} \qquad s = 1, 2, \ldots, M - 1,$$

and since $\lambda_s^- < -1$ for all $s$ the scheme is unstable for all values of $r$.

This example has shown us that the three-level explicit formula, whilst of the same local accuracy as the Crank–Nicolson scheme, and easier to use because it is explicit, fails because of instability. There are, however, successful formulas which are accurate, explicit and stable as we ask you to show in SAQ 2.

*SAQ 2*

Use the matrix method to investigate the stability of the explicit Du Fort and Frankel scheme,

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{u_{i+1,j} - u_{i,j+1} - u_{i,j-1} + u_{i-1,j}}{h^2},$$

applied to the problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad 0 < x < 1, \qquad t > 0,$$

$$U(0, t) = U(1, t) = 0 \qquad t \geqslant 0.$$

(Solution on p. 24.)

In SAQ 7 of *Unit 8* we showed that the local truncation error of the Du Fort and Frankel scheme applied to the simple diffusion equation is $O(k^2) + O(h^2) + O(k^2/h^2)$, and that the scheme is consistent with the diffusion equation provided

$$\lim_{h,k \to 0} \frac{k}{h} = 0.$$

Since our last SAQ has shown the method is stable, it is also convergent (by Lax's Theorem), and we can now ask whether it is better than the Crank–Nicolson scheme. Since the local order of accuracy of the Crank–Nicolson method is $O(h^2) + O(k^2)$ we cannot compare the two schemes directly because of the term $O(k^2/h^2)$ in the local truncation error of the Du Fort and Frankel scheme. The following SAQs provide some insight into the manner of resolving this problem.

*SAQ 3*

(a) Find the *maximum* order of accuracy of the Du Fort and Frankel scheme.

HINT: Set $k = O(h^p)$ as $h \rightsquigarrow 0$, where $p$ is some real number, and plot the order of accuracy as a function of $p$.

(b) Show that $p = 2$ is the most beneficial choice in (a). What does this result imply when choosing suitable values for the mesh ratio $r = k/h^2$?

(Solution on p. 26.)

*SAQ 4*

By setting $k = O(h^p)$ as $h \rightsquigarrow 0$ in the formula for the local order of accuracy of the Crank–Nicolson scheme, determine the most suitable value for $p$.

(Solution on p. 26.)

The two previous SAQs have shown us that optimum use is made of the Du Fort and Frankel scheme if we choose $k = O(h^2)$ whereas the Crank–Nicolson scheme is best when used with $k = O(h)$. For such optimum use both schemes have a local order of accuracy equal to $O(h^2)$. Therefore, if we consider $M - 1$ internal mesh points along each time level, so that $h = 1/M$, the step length $k$ for the Du Fort and Frankel scheme will have to be of the order of $1/M^2$. Therefore, to step forward one unit in time with the Du Fort and Frankel scheme requires something like $M^2$ steps. A similar calculation with the Crank–Nicolson scheme shows that only about $M$ steps are required for the same advance in time.

In order to compare these two schemes fairly we need to find the amount of work required by each scheme to evaluate a solution, along some fixed time level, with comparable accuracy. We have just seen that more steps are required with the Du Fort and Frankel scheme than with the Crank–Nicolson scheme to reach a fixed time level with the same local order of accuracy. However, the Crank–Nicolson scheme is implicit and requires more computation at each step. To begin we need to estimate the amount of computation required by each scheme in advancing one step in time. Writing the Du Fort and Frankel scheme in the form

$$u_{i,j+1} = u_{i,j-1} + \frac{a}{1+a}(u_{i+1,j} - u_{i,j-1} - u_{i,j-1} + u_{i-1,j}),$$

where $a = 2k/h^2$, we see that to evaluate $u_{i,j+1}$ we need to perform one multiplication and four additions or subtractions. Therefore, the calculation of a complete level of values, assuming $M - 1$ internal mesh points along each time level, involves $M - 1$ multiplications and $4M - 4$ additions or subtractions. The Crank–Nicolson scheme given by

$$-ru_{i-1,j+1} + (2 + 2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + (2 - 2r)u_{i,j} + ru_{i+1,j}$$

gives rise to a system of linear equations which can be written in the following form, assuming $M - 1$ internal mesh points along each time level:

$$\begin{bmatrix} b_1 & -c_1 & & & & \\ -a_2 & b_2 & -c_2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & -a_{M-2} & b_{M-2} & -c_{M-2} & \\ & & & -a_{M-1} & b_{M-1} & \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \cdot \\ \cdot \\ u_{M-2,j+1} \\ u_{M-1,j+1} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_{M-2} \\ d_{M-1} \end{bmatrix},$$

where $b_i = 2 + 2r$, $a_i = c_i = r$ and

$$d_i = ru_{i-1,j} + (2 - 2r)u_{i,j} + ru_{i+1,j}.$$

These equations can be solved by the recurrence relation form of Gauss elimination. Referring to *S: pages 21 and 22*, we can see that this method of solution will involve

$5M - 9$ multiplications or divisions and $3M - 6$ additions or subtractions (notice that we need only evaluate the $a_i/\alpha_{i-1}$ once). The calculation of the $d_i$ requires $2M - 2$ additions and $2M$ multiplications. (Note that, for example,

$$d_i = ru_{i-1,j} + (2 - 2r)u_{i,j} + ru_{i+1,j}$$

and

$$d_{i+2} = ru_{i+1,j} + (2 - 2r)u_{i+2,j} + ru_{i+3,j};$$

the term $ru_{i+1,j}$ occurs in both and need only be calculated once.) The amount of work required by the Crank - Nicolson scheme in stepping forward one level in time is therefore $7M - 9$ multiplications or divisions and $5M - 8$ additions or subtractions.

In performing $M^2$ steps with the Du Fort and Frankel scheme (to step forward one unit in time), we shall require $(M - 1)M^2$ multiplications and $4(M - 1)M^2$ additions. To reach the same time level with the Crank–Nicolson formula it takes of the order of $M$ steps with a total of $(7M - 9)M$ multiplications or divisions and $(5M - 8)M$ additions or subtractions.

To compare these figures we can assume that a modern computer takes the same time to compute a multiplication as it does a division and that additions take the same time as subtractions. Furthermore, for our present purposes, we shall assume that it takes ten times as long to compute a multiplication as it does an addition. With these assumptions we can easily deduce that for $M \geqslant 5$ (approximately) the Crank–Nicolson scheme is more efficient than the Du Fort and Frankel scheme.

*SAQ 5*

Compare the Crank–Nicolson and Du Fort and Frankel schemes for the solution of

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad 0 < x < 1, \qquad t > 0,$$

$$U(x, 0) = \sin \pi x \qquad 0 \leqslant x \leqslant 1,$$

$$U(0, t) = U(1, t) = 0 \qquad t \geqslant 0,$$

by obtaining values of $u$ along the level $t = 1$. Use several values of $h$ with the corresponding optimum values of $k$ obtained in SAQs 3 and 4.

Show that the choice of $k = h$ in the Du Fort and Frankel scheme results in a large error. How do you account for this?

(Solution on p. 27.)

It is also possible to construct implicit schemes, which do not have the consistency problem of the Du Fort and Frankel scheme (SAQ 7 of *Unit 8*). For example, we can construct a three-level implicit scheme in the manner in which the two-level Crank–Nicolson method was created. That is, we replace the second-order derivative $\partial^2 U/\partial x^2$ by the mean of its central difference representations on the three levels $j + 1, j$ and $j - 1$. The three-level, Crank–Nicolson type, implicit replacement of

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

is

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{1}{3}\left(\frac{\delta_x^2 u_{i,j+1} + \delta_x^2 u_{i,j} + \delta_x^2 u_{i,j-1}}{h^2}\right)$$

or

$$(1 - \tfrac{2}{3}r\delta_x^2)u_{i,j+1} = \tfrac{2}{3}r\delta_x^2 u_{i,j} + (1 + \tfrac{2}{3}r\delta_x^2)u_{i,j-1}. \qquad (5)$$

This scheme is stable for all values of $r$ and has a local order of accuracy equal to $O(h^2) + O(k^2)$.

When a three-level scheme is used, initial data must be provided on levels $j = 0$ and $j = 1$ in order to start the computation. Since a properly posed initial-boundary

value problem of parabolic type provides initial data along $t = 0$ only, we need to generate values along $t = k$ (i.e. $j = 1$) by some other means before we can use the three-level scheme. We could use a two-level difference formula of comparable accuracy to the three-level scheme or alternatively look for some analytical technique.

*SAQ 6*

Which scheme would you prefer to use to solve the simple diffusion equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2};$$

(a) the two-level Crank–Nicolson scheme.

or

(b) the three-level implicit scheme given by Equation (5)?

(Solution on p. 28.)

*SAQ 7*

A three-level implicit scheme for the diffusion equation, which has the same local order of accuracy as the Crank–Nicolson scheme and is unconditionally stable, is given by

$$\frac{3}{2}\left(\frac{u_{i,j+1} - u_{i,j}}{k}\right) - \frac{1}{2}\left(\frac{u_{i,j} - u_{i,j-1}}{k}\right) = \frac{1}{h^2}\,\delta_x^2 u_{i,j+1}.$$

Draw its molecular diagram. Show that this scheme involves less computational effort than the Crank–Nicolson scheme.

(Solution on p. 28.)

It is possible to construct (implicit) three-level schemes of higher order of accuracy which are always stable. However, such schemes involve more mesh points (typically up to nine points) and hence require more computational effort.

At first sight it would appear that three-level schemes have little more to offer us in the way of efficiency than two-level schemes, particularly since we have (usually) to use a two-level scheme to start them off. This is certainly true with linear equations. However, as we shall see in Section 15.1.4, three-level schemes can be of use in the solution of nonlinear equations. Before we go on to this topic we shall point out some of the difficulties associated with the numerical solution of nonlinear partial differential equations.

### 15.1.2 Nonlinear Equations

We have previously motivated our study of finite-difference methods by saying that at the present time it is usually only by numerical methods that we can effectively tackle *nonlinear* partial differential equations. The aim of this section is to present an introduction to this topic. We shall see that three-level schemes which we discussed in the previous section are of importance here. However, unlike the linear case, there is no unified theory for the solution of nonlinear partial differential equations. It turns out, however, that many techniques developed for linear equations apply equally well to nonlinear cases.

Here we shall deal only with particular initial-boundary value problems of "parabolic" type given by

$$\frac{\partial U}{\partial t} = f\left(x, t, U, \frac{\partial U}{\partial x}, \frac{\partial^2 U}{\partial x^2}\right) \qquad 0 < x < 1, \qquad 0 < t \leqslant T \tag{1}$$

where $f$ is some suitably well behaved function, together with (possibly) nonlinear boundary conditions.† When this problem is subject to smooth initial and boundary conditions it is properly posed in the region $0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T$ provided that

$$\frac{\partial f}{\partial U_{xx}} \geqslant a \tag{2}$$

for some constant $a > 0$. (The notation $\partial f/\partial U_{xx}$ represents the first partial derivative of the function $f$ with respect to its fifth argument, $\partial^2 U/\partial x^2$.)

Notice that when the right-hand side is just $\partial^2 U/\partial x^2$ so that Equation (1) reduces to the simple diffusion equation, we have

$$\frac{\partial f}{\partial U_{xx}} = 1$$

and Equation (2) is clearly satisfied.

Many of the numerical methods and techniques of proof for linear equations with constant coefficients carry over to nonlinear equations. However, in general, both the numerical processes and questions of stability and convergence are more complicated. The latter questions are largely unsolved and we generally replace "stability" by "local stability" on the assumption that the true solution of the differential equation varies little over small regions. We shall illustrate this point by investigating the stability in a simple example.

For demonstration purposes we consider the nonlinear equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 (U^m)}{\partial x^2} \tag{3}$$

where $m$ is some integer greater than 1. The simplest explicit scheme for this equation is given by

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j}^m - 2u_{i,j}^m + u_{i-1,j}^m}{h^2}. \tag{4}$$

The investigation of stability involves finding out how any errors in the pivotal values along one time level propagate to the next level. Suppose now that we perform the computation of Equation (4) with the values $u_{i,j}^*$ instead of $u_{i,j}$, where we have introduced an error $E_{i,j}$ into $u_{i,j}$ given by

$$E_{i,j} = u_{i,j}^* - u_{i,j}.$$

† The analytical treatment of nonlinear partial differential equations does not form part of this course; we shall not pursue further the properties that the function $f$ should possess to make the problem tractable. We feel obliged to point out, however, that considerable difficulties can arise in dealing with nonlinear differential equations and great care must be exercised.

That is, we evaluate an approximation to $u_{i,j+1}$ by the formula

$$\frac{u^*_{i,j+1} - u^*_{i,j}}{k} = \frac{(u^*_{i+1,j})^m - 2(u^*_{i,j})^m + (u^*_{i-1,j})^m}{h^2}. \tag{5}$$

Subtracting Equation (4) from Equation (5) and putting $r = k/h^2$, we obtain

$$(u^*_{i,j+1} - u_{i,j+1}) - (u^*_{i,j} - u_{i,j}) = r\{[(u^*_{i+1,j})^m - u^m_{i+1,j}] - 2[(u^*_{i,j})^m - u^m_{i,j}] + [(u^*_{i-1,j})^m - u^m_{i-1,j}]\}. \tag{6}$$

Now, if we assume that $E_{i,j}$ is sufficiently small (i.e. $E^m_{i,j} \simeq 0$ for $m \geqslant 2$), we can drop terms of second order and above in the binomial expansion, and obtain

$$(u^*_{i,j})^m - u^m_{i,j} = (u_{i,j} + E_{i,j})^m - u^m_{i,j} \simeq u^m_{i,j} + mE_{i,j}u^{m-1}_{i,j} - u^m_{i,j} = mE_{i,j}u^{m-1}_{i,j}.$$

We can obtain similar expressions for $(u^*_{i+1,j})^m - u^m_{i+1,j}$ and $(u^*_{i-1,j})^m - u^m_{i-1,j}$ so that Equation (6) becomes

$$E_{i,j+1} - E_{i,j} \simeq mr(u^{m-1}_{i+1,j}E_{i+1,j} - 2u^{m-1}_{i,j}E_{i,j} + u^{m-1}_{i-1,j}E_{i-1,j}). \tag{7}$$

The assumption that $u$ varies little over a small region means that $u_{i+1,j} \simeq u_{i,j} \simeq u_{i-1,j}$ and so Equation (7) gives

$$E_{i,j+1} \simeq E_{i,j} + mru^{m-1}_{i,j}(E_{i+1,j} - 2E_{i,j} + E_{i-1,j}).$$

We now apply von Neumann's method for investigating stability to this equation; we substitute into the equation the identity

$$E_{p,q} = e^{i\beta ph\xi q}, \qquad (i = \sqrt{-1})$$

which yields the conclusion, eventually, that the process is stable if

$$mru^{m-1}_{i,j} \leqslant \tfrac{1}{2}, \tag{8}$$

provided that $u^{m-1}_{i,j} \geqslant 0$. (The scheme is unconditionally unstable if $u^{m-1}_{i,j} < 0$.) This elementary analysis suggests that, for nonlinear problems, stability depends not only on the mesh ratio $r$ but also upon the solution being obtained. (Recall that, in the case of the (linear) one-dimensional heat equation the condition $r \leqslant \frac{1}{2}$ is necessary for stability, as we showed in Section 8.4 of *Unit 8*.)

In practice, when solving nonlinear partial differential equations, we keep a constant check on stability (if the difference scheme is not unconditionally stable) by making tests, using a condition such as inequality (8), and either stop when the condition is not satisfied or automatically alter $k$ to restore stability.

Returning to Equation (1), a satisfactory explicit finite-difference replacement can be written as

$$u_{i,j+1} = u_{i,j} + kf\left(ih, jk, u_{i,j}, \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \frac{1}{h^2}\delta^2_x u_{i,j}\right).$$

This approximation is very simple to use, but suffers from the disadvantages that the mesh ratio $r$ is strictly limited for stability.

As in the linear case, the stability restriction can usually be removed by turning to implicit schemes of the Crank–Nicolson type. For Equation (1), the Crank–Nicolson equivalent scheme is

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2}(f_{j+1} + f_j), \tag{9}$$

with obvious notation.

Unfortunately, depending upon the form of $f$, the algebraic problem of determining $u_{i,j+1}$ from Equation (9) may become quite complicated, since, if $f$ is nonlinear, the algebraic equations at each time level are usually nonlinear. We may still be able to solve these equations however, as we shall illustrate in the following section.

*SAQ 8*

Making the same assumptions as in the text show that the Crank–Nicolson type formula (9) is unconditionally stable locally when applied to Equation (3). (Use the von Neumann method and assume $u_{i,j}^{m-1} \geqslant 0$.)

(Solution on p. 29.)

### 15.1.3 Iterative Solution of Nonlinear Algebraic Equations

In most cases it is not possible to give an expression in closed form for the solutions of systems of nonlinear algebraic equations. Even when such expressions exist they may not be very useful for numerical purposes. As is frequently the case in numerical analysis, iterative methods which lead to approximations for the solutions are the best tools for this problem. There exist a large number of different methods, some better than others, but no one method is best in all cases. We shall briefly look at one such method which is often used, namely *Newton's method*.

In *Unit M100 14, Sequences and Limits II* Newton's method for solving the single equation

$$f(x) = 0$$

was presented. It is the iterative scheme given by

$$x^{(n+1)} = x^{(n)} - \frac{f(x^{(n)})}{f'(x^{(n)})} \qquad n = 0, 1, 2, \ldots .$$

If the initial iterate $x^{(0)}$ is sufficiently close to a root $\alpha$ of $f(x) = 0$ then the successive iterates $x^{(1)}, x^{(2)}, \ldots$ converge to $\alpha$.

Let us now look at the case of two equations in two unknowns given by

$$f_1(x_1, x_2) = 0$$

$$f_2(x_1, x_2) = 0.$$

Suppose that we have initial approximations $x_1^{(0)}$ and $x_2^{(0)}$ to the roots $\alpha_1$ and $\alpha_2$ such that

$$\alpha_1 = x_1^{(0)} + \delta x_1$$

$$\alpha_2 = x_2^{(0)} + \delta x_2.$$

Then first-order Taylor approximations (*Unit M201 14, Bilinear Forms*) give the relations

$$f_1(\alpha_1, \alpha_2) = f_1(x_1^{(0)} + \delta x_1, x_2^{(0)} + \delta x_2)$$

$$\simeq f_1(x_1^{(0)}, x_2^{(0)}) + \frac{\partial f_1}{\partial x_1}(x_1^{(0)}, x_2^{(0)})\delta x_1 + \frac{\partial f_1}{\partial x_2}(x_1^{(0)}, x_2^{(0)})\delta x_2$$

and

$$f_2(\alpha_1, \alpha_2) = f_2(x_1^{(0)} + \delta x_1, x_2^{(0)} + \delta x_2)$$

$$\simeq f_2(x_1^{(0)}, x_2^{(0)}) + \frac{\partial f_2}{\partial x_1}(x_1^{(0)}, x_2^{(0)})\delta x_1 + \frac{\partial f_2}{\partial x_2}(x_1^{(0)}, x_2^{(0)})\delta x_2.$$

Setting these expressions equal to zero, for a solution, we can write them in matrix form as

$$\begin{bmatrix} f_1(x_1^{(0)}, x_2^{(0)}) \\ f_2(x_1^{(0)}, x_2^{(0)}) \end{bmatrix} + \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where the derivatives are evaluated at $(x_1^{(0)}, x_2^{(0)})$. We see that the correction vector $(\delta x_1, \delta x_2)$ is given by

$$\begin{bmatrix} \delta x_1 \\ \\ \delta x_2 \end{bmatrix} \simeq -J^{-1}_{(x_1^{(0)}, x_2^{(0)})} \begin{bmatrix} f_1(x_1^{(0)}, x_2^{(0)}) \\ \\ f_2(x_1^{(0)}, x_2^{(0)}) \end{bmatrix}$$

where

$$J = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}.$$

The notation $J^{-1}_{(x_1, x_2)}$ means that the matrix $J^{-1}$ is evaluated using the values $x_1$ and $x_2$. The iterative scheme is easily seen to be given by

$$\begin{bmatrix} x_1^{(n+1)} \\ \\ x_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} x_1^{(n)} \\ \\ x_2^{(n)} \end{bmatrix} - J^{-1}_{(x_1^{(n)}, x_2^{(n)})} \begin{bmatrix} f_1(x_1^{(n)}, x_2^{(n)}) \\ \\ f_2(x_1^{(n)}, x_2^{(n)}) \end{bmatrix} \qquad n = 0, 1, 2, \ldots .$$

In the general case we want to solve a system of equations of the form

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}, \tag{1}$$

where $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_N(\mathbf{x}))$ and $\mathbf{x}$ is the $N$-tuple $(x_1, x_2, \ldots, x_N)$. Equation (1) can be expanded using the first-order Taylor approximation to give the iterative scheme

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} - J^{-1}_{\mathbf{x}^{(n)}} \mathbf{f}(\mathbf{x}^{(n)}) \tag{2}$$

where $J$ is now given by

$$J_{\mathbf{x}} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(\mathbf{x}) & \dfrac{\partial f_1}{\partial x_2}(\mathbf{x}) \cdots \dfrac{\partial f_1}{\partial x_N}(\mathbf{x}) \\ \dfrac{\partial f_2}{\partial x_1}(\mathbf{x}) & \dfrac{\partial f_2}{\partial x_2}(\mathbf{x}) \cdots \dfrac{\partial f_2}{\partial x_N}(\mathbf{x}) \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \dfrac{\partial f_N}{\partial x_1}(\mathbf{x}) & \dfrac{\partial f_N}{\partial x_2}(\mathbf{x}) \cdots \dfrac{\partial f_N}{\partial x_N}(\mathbf{x}) \end{bmatrix}.$$

which is a direct generalisation of the two-equation case. The *determinant* of the matrix $J_{\mathbf{x}}$ is known as the **Jacobian** of the functions $\{f_i\}$ at $\mathbf{x}$. Throughout this discussion we have assumed that the inverse $J^{-1}$ exists, which is the same thing as saying that the Jacobian, $\det[J_{\mathbf{x}}]$, is nonzero in a neighbourhood of the root $\mathbf{x} = \boldsymbol{\alpha}$ of Equation (1). Actually this ensures that we can find a neighbourhood of $\mathbf{x} = \boldsymbol{\alpha}$ in which $\boldsymbol{\alpha}$ is the only solution.

In using the scheme (2), an inverse need not be *computed* at each iteration; instead a linear system of order $N$ has to be solved. We can see this by noting that Equation (2) can be written in the form

$$J_{\mathbf{x}^{(n)}}(\mathbf{x}^{(n)} - \mathbf{x}^{(n+1)}) = \mathbf{f}(\mathbf{x}^{(n)})$$

which is a linear system which can be solved for the vector $\mathbf{x}^{(n)} - \mathbf{x}^{(n+1)}$.

Although we shall not investigate this method any further it must be stated that the choice of initial guess is important with Newton's method because if the approximation is not close to the solution, the iteration will not converge in general. Fortunately in our problem we can often get a good initial approximation by extrapolation from previous levels.

**Example**

Evaluate the finite-difference solution, along the first time level, of the nonlinear problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U^2}{\partial x^2} \qquad 0 < x < 1, \qquad t > 0,$$

$$U(x, 0) = \sin \pi x \qquad 0 \leqslant x \leqslant 1,$$

$$U(0, t) = U(1, t) = 0 \qquad t \geqslant 0,$$

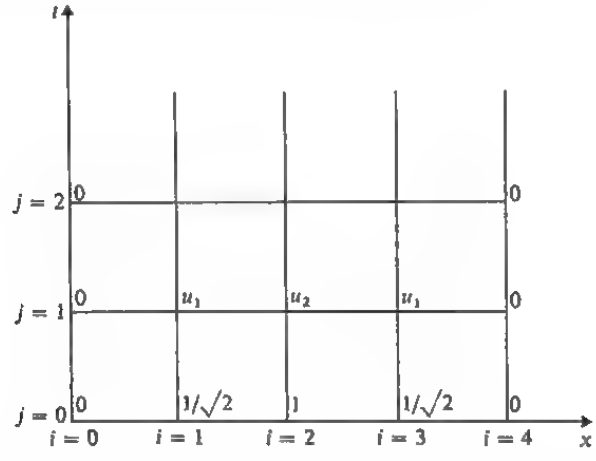
taking $r = k/h^2 = 1$ and $h = \frac{1}{4}$ in the implicit scheme given by

$$u_{i,j+1} - u_{i,j} = \frac{k}{2h^2}(u^2_{i+1,j+1} - 2u^2_{i,j+1} + u^2_{i-1,j+1} + u^2_{i+1,j} - 2u^2_{i,j} + u^2_{i-1,j}),$$

and solving the nonlinear equations by Newton's method.

*Solution*

The region of interest is shown in the following diagram.




There are three unknowns along the first time level, $u_{1,1}$, $u_{2,1}$ and $u_{3,1}$. By symmetry $u_{1,1} = u_{3,1}$ so we need only consider $u_{1,1}$ and $u_{2,1}$ which we shall write as $u_1$ and $u_2$ respectively. Substituting the initial and boundary conditions into the difference equation yields two equations for $u_1$ and $u_2$ given by

$$f_1(u_1, u_2) \equiv -2u_1^2 + u_2^2 - 2u_1 + \sqrt{2} = 0,$$

$$f_2(u_1, u_2) \equiv 2u_1^2 - 2u_2^2 - 2u_2 + 1 = 0.$$

Using the form of Newton's method given by

$$J_{\mathbf{u}^{(n)}}(\mathbf{u}^{(n)} - \mathbf{u}^{(n+1)}) = \mathbf{f}(\mathbf{u}^{(n)}),$$

where $\mathbf{u} = (u_1, u_2)$, we have

$$\begin{bmatrix} -4u_1^{(n)} - 2 & 2u_2^{(n)} \\ 4u_1^{(n)} & -4u_2^{(n)} - 2 \end{bmatrix} \begin{bmatrix} u_1^{(n)} - u_1^{(n+1)} \\ u_2^{(n)} - u_2^{(n+1)} \end{bmatrix} = \begin{bmatrix} -2(u_1^{(n)})^2 + (u_2^{(n)})^2 - 2u_1^{(n)} + \sqrt{2} \\ 2(u_1^{(n)})^2 - 2(u_2^{(n)})^2 - 2u_2^{(n)} + 1 \end{bmatrix},$$

since

$$\frac{\partial f_1}{\partial u_1} = -4u_1 - 2, \quad \frac{\partial f_1}{\partial u_2} = 2u_2, \quad \frac{\partial f_2}{\partial u_1} = 4u_1, \quad \frac{\partial f_2}{\partial u_2} = -4u_2 - 2.$$

Starting the iteration with $(u_1^{(0)}, u_2^{(0)}) = (u_{1,0}, u_{2,0}) = (\frac{1}{2}\sqrt{2}, 1)$ we get

$$\begin{bmatrix} -4.8284 & 2.0 \\ 2.9284 & -6.0 \end{bmatrix} \begin{bmatrix} 0.7071 - u_1^{(1)} \\ 1.0000 - u_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

from which $(u_1^{(1)}, u_2^{(1)}) = (0.5355, 0.5857)$. The second iteration yields

$$\begin{bmatrix} -4.1420 & 1.1714 \\ 2.1420 & -4.3428 \end{bmatrix} \begin{bmatrix} 0.5355 - u_1^{(2)} \\ 0.5857 - u_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0.1127 \\ -0.2840 \end{bmatrix}$$

and so $(u_1^{(2)}, u_2^{(2)}) = (0.5456, 0.5253)$. The third iteration gives

$$\begin{bmatrix} -4.1826 & 1.0506 \\ 2.1826 & -4.1012 \end{bmatrix} \begin{bmatrix} 0.5456 - u_1^{(3)} \\ 0.5253 - u_2^{(3)} \end{bmatrix} = \begin{bmatrix} 0.003438 \\ -0.007075 \end{bmatrix}$$

from which $(u_1^{(3)}, u_2^{(3)}) = (0.5461, 0.5258)$. This final iteration leaves residuals $(-7.622 \times 10^{-6}, 1.379 \times 10^{-5})$. Hence,

$$u_{1,1} = u_{3,1} = 0.546,$$

$$u_{2,1} = 0.524.$$

*SAQ 9*

Evaluate the finite-difference solution along the first time level of the nonlinear problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U^2}{\partial x^2} \qquad 0 < x < 1, \quad t > 0,$$

$$U(x, 0) = 16x(1 - x) + 1 \qquad 0 \leqslant x \leqslant 1,$$

$$U(0, t) = U(1, t) = 1 \qquad t \geqslant 0,$$

Taking $r = \frac{1}{10}$ and $h = \frac{1}{4}$ in the Crank–Nicolson type formula quoted in the example above. Perform only the first three iterations using Newton's method.

(Solution on p. 30.)

### 15.1.4 Three-Level Schemes for Nonlinear Equations

We have seen that finite-difference methods based on two time levels for both linear and nonlinear equations have poor stability properties if explicit, and are difficult (complicated) to use if implicit. Therefore, we turn to three-level schemes which, in the linear case, can combine good accuracy with good stability properties although they are not very economic in computation time. We shall now proceed to investigate the nonlinear parabolic equation given by

$$b(U)\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(a(U)\frac{\partial U}{\partial x}\right) \quad \text{with } a(U) > 0, b(U) > 0. \tag{1}$$

The simplest three-level explicit approximation to this equation is given by

$$b(u_{i,j})\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{1}{h^2}\delta_x\{a(u_{i,j})\delta_x u_{i,j}\}. \tag{2}$$

There seems little point in pursuing this scheme further because we have already seen in the example in Section 15.1.1 that when $a(U) = b(U) = 1$ it is completely unstable. However, in the linear case we obtained unconditional stability by using the Crank–Nicolson device. This fact indicates how we can proceed here. Expanding Equation (2) gives

$$b(u_{i,j})(u_{i,j+1} - u_{i,j-1}) = 2r\{a(u_{i+\frac{1}{2},j})(u_{i+1,j} - u_{i,j}) - a(u_{i-\frac{1}{2},j})(u_{i,j} - u_{i-1,j})\}$$

where we have used the central-difference formula

$$\delta_x u_{i,j} = u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}.$$

We now replace $u_{i+1,j}$, $u_{i,j}$ and $u_{i-1,j}$ by their averages over three time levels $j-1$, $j$ and $j+1$, so that

$u_{i+1,j}$ is replaced by $\frac{1}{3}(u_{i+1,j+1} + u_{i+1,j} + u_{i+1,j-1})$,

$u_{i,j}$ is replaced by $\frac{1}{3}(u_{i,j+1} + u_{i,j} + u_{i,j-1})$

and

$u_{i-1,j}$ is replaced by $\frac{1}{3}(u_{i-1,j+1} + u_{i-1,j} + u_{i-1,j-1})$;

and we replace $u_{i\pm\frac{1}{2},j}$ by an average over neighbouring mesh points so that $a(u_{i+\frac{1}{2},j})$ becomes

$$\alpha^+_{i,j} = a\left(\frac{u_{i+1,j} + u_{i,j}}{2}\right)$$

and $a(u_{i-\frac{1}{2},j})$ becomes

$$\alpha^-_{i,j} = a\left(\frac{u_{i,j} + u_{i-1,j}}{2}\right).$$

This process leads to the finite-difference scheme given by

$$h(u_{i,j})(u_{i,j+1} - u_{i,j-1}) = \tfrac{2}{3}r[\alpha^+_{i,j}(u_{i+1,j+1} - u_{i,j+1} + u_{i+1,j} - u_{i,j} + u_{i+1,j-1} - u_{i,j-1}) - \alpha^-_{i,j}(u_{i,j+1} - u_{i-1,j+1} + u_{i,j} - u_{i-1,j} + u_{i,j-1} - u_{i-1,j-1})].$$

This replacement does not alter the order of accuracy which remains $O(h^2) + O(k^2)$. Note that for nonlinear equations Lax's Theorem cannot be used and we have to aim directly at a proof of convergence. In fact, it has been shown that this scheme is both stable and convergent for all values of $r$. The most remarkable feature of the scheme is that, although implicit, it gives rise to a system of linear algebraic equations to be solved at each time level. Hence, the above scheme avoids the complication of solving sets of nonlinear equations inherent in the Crank–Nicolson method on two time levels!

The following SAQ looks at the accuracy of the above scheme. The solution is quite involved since you will need to expand $\alpha^+_{i,j}$ and $\alpha^-_{i,j}$ by Taylor's Theorem.

*SAQ 10*

Show that the local order of accuracy of the above three-level scheme when applied to the nonlinear equation (1) is $O(h^2) + O(k^2)$.

(Solution on p. 30.)

## 15.2 PARABOLIC EQUATIONS IN TWO SPACE DIMENSIONS

A typical parabolic equation in two space dimensions is the two-dimensional heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \qquad t > 0, \quad x, y \in (0, 1). \tag{1}$$

We shall examine the numerical solution of this equation in a square region of the $xy$-plane in the usual way by covering the region with a rectangular mesh with spacing $h$ in both the $x$- and $y$- directions. The mesh points $(x_i, y_j, t_n)$ are given by $x_i = ih$, $y_j = jh$ and $t_n = nk$ where $i$, $j$ and $n$ are integers and $i = j = n = 0$ is the origin. The solution at $(x_i, y_j, t_n)$ of a given finite-difference scheme will be denoted by $u_{i,j,n}$, and the true solution of the differential equation at that point will be denoted by $U(x_i, y_j, t_n)$ or more simply, by $U_{i,j,n}$.

The standard explicit finite-difference replacement of Equation (1) can be obtained by replacing the time derivative by the usual forward-difference formula and the two space derivatives by the usual central-difference formula. Thus, we obtain

$$\frac{1}{k}\Delta_t u_{i,j,n} = \frac{\mathrm{I}}{h^2}(\delta_x^2 + \delta_y^2)u_{i,j,n} \tag{2}$$

or

$$u_{i,j,n+1} = u_{i,j,n} + r(u_{i+1,j,n} + u_{i-1,j,n} + u_{i,j+1,n} + u_{i,j-1,n} - 4u_{i,j,n}), \tag{3}$$

where $r = k/h^2$ is the usual mesh ratio for parabolic equations.

*SAQ 11*

Find the local truncation error of formula (2).

(Solution on p. 32.)

The stability of a formula used to approximate equations in two or more space dimensions can be investigated by von Neumann's method which we discussed in Section 8.4.3 of *Unit 8, Stability*. In the case of Equation (1), where there are three independent variables, we look for separable solutions to the *error equation* which have the form

$$E_{p,q,n} = e^{i\beta_1 ph} e^{i\beta_2 qh} \xi^n.$$

The error equation is obtained by considering starting the computation with the vector $\mathbf{u}_0^*$ instead of $\mathbf{u}_0$ when $n = 0$ and putting

$$E_{p,q,n} = u_{p,q,n} - u_{p,q,n}^*.$$

The von Neumann criterion for stability is then

$$|\xi| \leqslant 1,$$

as is easily verified by the techniques of *Unit 8*.

*SAQ 12*

(a) Use von Neumann's method to investigate the stability of formula (3).

(b) Give a sufficient condition for the convergence of formula (3) when applied to the differential equation (1) as $h, k \rightsquigarrow 0$.

(Solution on p. 32.)

As usual, poor stability properties leading to increased computation mean that explicit methods are rarely used to solve initial-boundary value problems in two or more space dimensions; implicit methods are almost always used in preference. As we have seen repeatedly, implicit methods require sets of algebraic equations, which are linear if the differential equation is linear, to be solved at each new time level.

If we now apply the Crank–Nicolson averaging technique to Equation (2) we obtain the finite-difference scheme

$$(1 - \tfrac{1}{2}r\delta_x^2 - \tfrac{1}{2}r\delta_y^2)u_{i,j,n+1} = (1 + \tfrac{1}{2}r\delta_x^2 + \tfrac{1}{2}r\delta_y^2)u_{i,j,n}. \tag{4}$$

As we are dealing with two space dimensions, we effectively have to solve a boundary value problem at each time step. We can therefore exploit our knowledge of the numerical solution of such problems (gained from *Unit 11, Boundary Value Problems*) as the following SAQ illustrates.

*SAQ 13*

If $U$ is specified on the boundaries of a square region $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$, for $t \geqslant 0$, show that the SOR method, discussed in Section 11.4 of *Unit 11*, can be used to solve the system of equations arising from the use of Equation (4) to approximate the solution of the differential equation (1). Find the optimum relaxation factor when there are $M + 1$ mesh points in each of the $x$- and $y$- directions.

(Solution on p. 33.)

## 15.3 DISCONTINUITIES IN THE SUBSIDIARY CONDITIONS

The very nature of finite-difference approximations to partial differential equations means that they are unsatisfactory near discontinuities. We have usually derived finite-difference replacements to derivatives from first- or second-order Taylor approximations which we know to be of dubious value at or near discontinuities. All the estimates of local order of accuracy and global error for such schemes depend on the boundedness of partial derivatives of the true solution to the given partial differential equation. These estimates may not be valid if the solution has discontinuities at one or more boundary points. Obviously great care has to be exercised in these cases and usually we have to treat each case individually.

The most common form of discontinuity in initial-boundary value problems occurs at the corners of the solution domain, where the initial line meets a boundary line. If the initial line is $t = 0$ and the other boundaries are $x = 0$ and $x = 1$, typical initial and boundary conditions are given by

$$U(x, 0) = f_1(x), \quad U(0, t) = f_2(t), \quad U(1, t) = f_3(t),$$

which we usually take to mean

$$\lim_{t \to 0^+} U(x, t) = f_1(x) \qquad 0 < x < 1,$$

$$\lim_{x \to 0^+} U(x, t) = f_2(t) \qquad t > 0,$$

$$\lim_{x \to 1^-} U(x, t) = f_3(t) \qquad t > 0,$$

respectively. The corners (0, 0) and (0, 1) sometimes involve some form of discontinuity. The worst type of discontinuity is in the function value where, for example,

$$f_1(0) \neq f_2(0) \quad \text{or} \quad f_1(1) \neq f_3(0).$$

Weaker forms of discontinuity also exist. For example, suppose that we are considering the diffusion equation, with initial and boundary conditions of the above type, given by

$$f_1(x) = 0$$

$$f_2(t) = t.$$

Obviously $f_1(0) = f_2(0) = 0$ and there is no discontinuity in the function values at the point (0, 0). However

$$\lim_{t \to 0} \frac{\partial f_2}{\partial t} \neq \lim_{x \to 0} \frac{\partial^2 f_1}{\partial x^2}.$$

The effect of such discontinuities does not penetrate into the solution domain for the analytical solution. However, this is no longer true for finite-difference solutions which have doubtful value in the neighbourhood of the points of discontinuity. Usually we do not have to worry because the true solution becomes "smoother" as $t$ increases and any errors introduced in the finite-difference solution decay, provided the methods used are stable. We have seen a good example of this in Section 5.2 of *Unit 5, Initial Value Problems*. However, this is not always the case, as you will see in the next reading passage.

*READ S: page 86, line 18 to page 89, line −1.*

As a simple example of how analytical numerical methods can be used together to advantage we consider Exercise 2 on *S: page 46*. There is a discontinuity at the origin of the given problem which means that the finite-difference solution is inaccurate for small values of $x$ and $t$. (See the table in *S: page 48*.) If, however, we take the analytical solution at $t = 0.001$ (given in Table 2.28) as the starting point for the simple explicit scheme using $r = 0.1$ (given at the top of *S: page 47*) we obtain the results in the following table.

| | $x = 0$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|---|
| $t = 0.001$ | 0 | 0.9747 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.002 | 0 | 0.8798 | 0.9975 | 1.0000 | 1.0000 | 1.0000 |
| 0.003 | 0 | 0.8036 | 0.9860 | 0.9997 | 1.0000 | 1.0000 |
| 0.004 | 0 | 0.7415 | 0.9691 | 0.9984 | 1.0000 | 1.0000 |
| 0.005 | 0 | 0.6901 | 0.9493 | 0.9956 | 0.9998 | 1.0000 |

If we concentrate on the results at $x = 0.1$ where the finite difference results are poorest we find that, at $t = 0.002$, the percentage error was previously 7.5 per cent whereas now it is down to 0.7 per cent. Similarly, at $t = 0.005$ the percentage error has improved from 3.8 per cent to $-1.1$ per cent.

## 15.4 SUMMARY

The quest for simple schemes with good accuracy leads us to look at *three-level schemes*. We have seen how to adapt the matrix method to investigate the stability of these schemes.

We have investigated a simple class of nonlinear partial differential equations and showed how our idea of stability has to be amended to *local stability* before we can investigate our simple finite-difference schemes analytically. We have seen that, in general, implicit methods for nonlinear equations necessitate the solution of sets of nonlinear algebraic equations and we presented a straightforward method for their solution. However, this method illustrates the major difficulty in solving nonlinear algebraic equations, namely the large amount of computation which is necessary. Fortunately, for one simple class of problems, we were able to quote a three-level implicit scheme which gives rise to linear algebraic equations.

We have also looked briefly at the diffusion equation in two dimensions. Finally, we have taken a look at the problems associated with discontinuities in parabolic equations and made the important point that finite-difference methods should not be used in isolation. It is important that numerical methods should be used in conjunction with analytical techniques in order to obviate those difficulties which usually arise in real problems.

## 15.5 SOLUTIONS TO SELF-ASSESSMENT QUESTIONS

*Solution to SAQ 1*

The local truncation error of the three-level scheme is given by

$$T_{i,j} = \frac{U_{i,j+1} - U_{i,j-1}}{2k} - \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}$$

where $U$ is the true solution of the differential equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}.$$

Taylor's Theorem gives

$$U_{i,j+1} - U_{i,j-1} = 2k\frac{\partial U_{i,j}}{\partial t} + \frac{k^3}{3}\frac{\partial^3 U_{i,j}}{\partial t^3} + O(k^5)$$

and

$$U_{i+1,j} - 2U_{i,j} + U_{i-1,j} = h^2\frac{\partial^2 U_{i,j}}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 U_{i,j}}{\partial x^4} + O(h^6).$$

Hence,

$$T_{i,j} = O(k^2) + O(h^2)$$

since

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}.$$

Thus, the three-level explicit scheme is locally more accurate than the two-level explicit scheme, for which, as we saw in *Unit 5*,

$$T_{i,j} = O(k) + O(h^2).$$

*Solution to SAQ 2*

The Du Fort and Frankel scheme can be written as

$$(1 + 2r)u_{i,j+1} = 2r(u_{i+1,j} + u_{i-1,j}) + (1 - 2r)u_{i,j-1},$$

that is,

$$A\mathbf{u}_{j+1} = B\mathbf{u}_j + C\mathbf{u}_{j-1}$$

where $A = (1 + 2r)I$, $C = (1 - 2r)I$ and $B$ is given by

$$B = 2r\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

The square matrices $A$, $B$, $C$ and $I$ have order $M - 1$, where $M - 1$ is the number of internal mesh points along each time level. The equation

$$\det(\lambda^2 A - \lambda B - C) = 0$$

now becomes

$$\det\left[B - \frac{(1 + 2r)\lambda^2 - (1 - 2r)}{\lambda}I\right] = 0$$

and if $\mu$ is an eigenvalue of $B$ then $\lambda$ is an eigenvalue of the amplification matrix if it satisfies

$$\frac{(1 + 2r)\lambda^2 - (1 - 2r)}{\lambda} = \mu.$$

The eigenvalues of $B$ are given by (see Appendix to *Unit 8*)

$$\mu_s = 4r \cos\left(\frac{s\pi}{M}\right) \qquad s = 1, 2, \ldots, M - 1,$$

and hence

$$(1 + 2r)\lambda_s^2 - 4r \cos\left(\frac{s\pi}{M}\right) \lambda_s - (1 - 2r) = 0,$$

from which

$$\lambda_s^{\pm} = \frac{2r \cos\left(\frac{s\pi}{M}\right) \pm \left[1 - 4r^2 \sin^2\left(\frac{s\pi}{M}\right)\right]^{\frac{1}{2}}}{1 + 2r}.$$

For stability we require

$$\max_s |\lambda_s^{\pm}| \leqslant 1.$$

If $4r^2 \sin^2(s\pi/M) \leqslant 1$ this becomes

$$-(1 + 2r) \leqslant 2r \cos\left(\frac{s\pi}{M}\right) \pm \left[1 - 4r^2 \sin^2\left(\frac{s\pi}{M}\right)\right]^{\frac{1}{2}} \leqslant 1 + 2r,$$

which is clearly satisfied.

If $4r^2 \sin^2(s\pi/M) > 1$, we have $r > \frac{1}{2}$ and

$$\lambda_s^{\pm} = \frac{2r \cos\left(\frac{s\pi}{M}\right) \pm i\left[4r^2 \sin^2\left(\frac{s\pi}{M}\right) - 1\right]^{\frac{1}{2}}}{1 + 2r}, \qquad (i = \sqrt{-1})$$

so that

$$\begin{aligned} |\lambda_s^{\pm}|^2 &= \frac{4r^2 \cos^2\left(\frac{s\pi}{M}\right) + 4r^2 \sin^2\left(\frac{s\pi}{M}\right) - 1}{(1 + 2r)^2} \\ &= \frac{4r^2 - 1}{(1 + 2r)^2} \\ &= \frac{2r - 1}{1 + 2r}, \end{aligned}$$

and $|\lambda_s^{\pm}| \leqslant 1$ for all $r > \frac{1}{2}$.

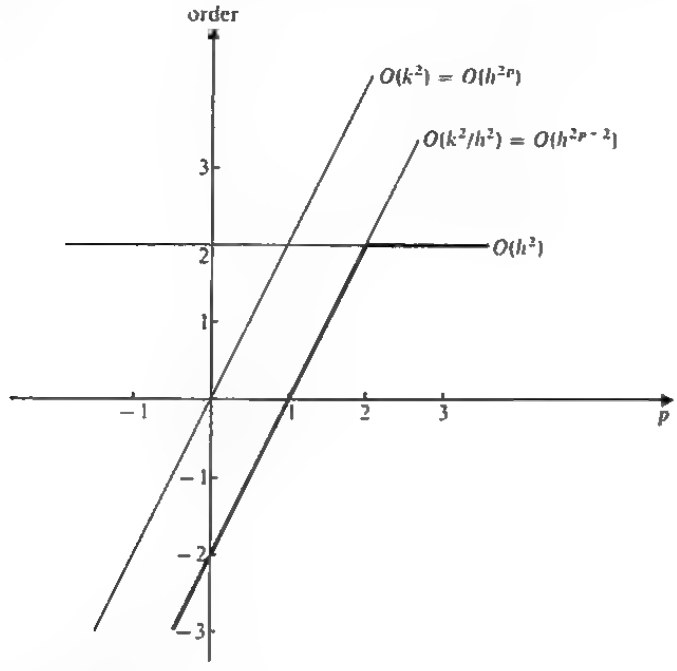
Hence, the scheme is stable for all (positive) values of $r$.

*Solution to SAQ 3*

(a) The order of accuracy of the Du Fort and Frankel scheme is

$$O(h^2) + O(k^2) + O(k^2/h^2);$$

we can plot the order of each term against $p$ where $k = O(h^p)$ as $h \rightsquigarrow 0$.



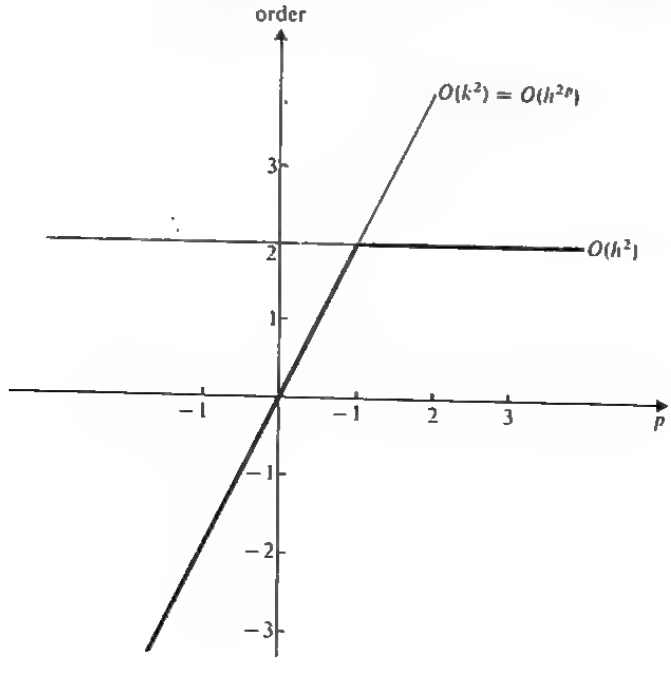

The order of the method is shown by the dark line. Thus the maximum order of the method is $O(h^2)$.

(b) We want to obtain the largest order in the method and at the same time achieve the largest step size $k$ (as $h \rightsquigarrow 0$). Now since $h$ is usually small and $k = O(h^p)$ we require $p$ to be the smallest value such that the order remains $O(h^2)$. That is, $p = 2$.

Notice that $p = 2$ means that $r = k/h^2 =$ constant. We prefer to have $r =$ constant in practice since this has the added advantage of giving convenient calculations.

*Solution to SAQ 4*

The local order of accuracy of the Crank–Nicolson formula is given by $O(h^2) + O(k^2)$. Putting $k = O(h^p)$ as $h \rightsquigarrow 0$ gives the following graph of order plotted against $p$.



The order of the method is shown by the dark line. Thus the maximum order of the method is 2. Since $h$ is small, the largest value of $k$ is obtained when $p = 1$. That is, we prefer to take $k = O(h)$.

*Solution to SAQ 5*

The following is a sample output from the program $COM321 which uses both the Crank–Nicolson and Du Fort and Frankel schemes to solve the given problem. The program has one input parameter $N$ which is the total number of mesh points along each time level. The results from the schemes are compared at $t = 1$ by printing out the errors (the difference between the computed solutions and the analytical solution). The output consists of two tables of numbers. The first is a comparison between the Crank–Nicolson scheme with $k = h$ and the Du Fort and Frankel scheme with $k = h^2$. The second table shows the errors of the Crank–Nicolson and the Du Fort and Frankel schemes both with $k = h$.

```
RUN
COM321

A PROGRAM TO COMPARE THE EFFICIENCIES OF THE CRANK-NICOLSON
AND DU FORT-FRANKEL SCHEMES


INPUT THE NUMBER OF MESH POINTS ALONG EACH TIME LEVEL(N)?7

SOLUTIONS EVALUATED AT T=1.Ø

CRANK - NICOLSON SCHEME WITH K=H (  6     TIME STEPS)
DU FORT - FRANKEL SCHEME WITH K=H↑2 (  36    TIME STEPS)

                        ERROR IN
  X                CRANK - N       DU FORT - F
 .166667          2.5Ø243E-Ø5     2.58457E-Ø5
 .333333          4.33536E-Ø5     4.4766ØE-Ø5
 .5               5.ØØ798E-Ø5     5.16914E-Ø5
 .666667          4.347Ø4E-Ø5     4.47661E-Ø5
 .833333          2.48565E-Ø5     2.58458E-Ø5

DU FORT - FRANKEL SCHEME WITH K=H ( 6     TIME STEPS)

                        ERROR IN
  X                CRANK - N       DU FORT - F
 .166667          2.5Ø243E-Ø5     .319499
 .333333          4.33536E-Ø5     .553387
 .5               5.ØØ798E-Ø5     .638996
 .666667          4.347Ø4E-Ø5     .553387
 .833333          2.48565E-Ø5     .319498

DONE
```

We have used a value of 7 for the input parameter $N$, giving $h = \frac{1}{6}$ for both schemes. The first table shows that, as predicted in the text, the Du Fort and Frankel scheme with $k = h^2$ gives an accuracy comparable with that obtained using the Crank-Nicolson scheme with $k = h$. [With $N$ larger than about 7 it will be found that there is a considerable waiting time between the computer printing the message

DU FORT – FRANKEL SCHEME WITH K = H↑2...

and the table which follows it. This time is mostly taken up by the computer evaluating the Du Fort and Frankel scheme. There is no noticeable time taken in computing the Crank- Nicolson solution.]

The second table, in which $k = h$ for both schemes shows clearly the inferiority of the Du Fort and Frankel scheme; the errors do appear to be extremely large. We can account for this by reference to the expression for the local truncation error of the Du Fort and Frankel scheme,

$$T_{i,j} = \frac{k^2}{6}\frac{\partial^3 U_{i,j}}{\partial t^3} - \frac{h^2}{12}\frac{\partial^4 U_{i,j}}{\partial x^4} + \frac{k^2}{h^2}\frac{\partial^2 U_{i,j}}{\partial t^2} + \cdots$$

where we see that the ratio $k^2/h^2$ is unity when $k = h$. This means that the local error will contain large contributions from the term $\partial^2 U_{i,j}/\partial t^2$ and it is likely that the computed solution is a better approximation to the hyperbolic equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial t^2} = 0$$

than to the given parabolic equation. (See SAQ 7 of *Unit 8*.)

*Solution to SAQ 6*

In full the three-level implicit scheme is

$$-\frac{2r}{3}u_{i+1,j+1} + \left(1 + \frac{4r}{3}\right)u_{i,j+1} - \frac{2r}{3}u_{i-1,j+1} = \frac{2r}{3}u_{i+1,j} - \frac{4r}{3}u_{i,j} + \frac{2r}{3}u_{i-1,j} + \frac{2r}{3}u_{i+1,j-1} + \left(1 - \frac{4r}{3}\right)u_{i,j-1} + \frac{2r}{3}u_{i-1,j-1}.$$

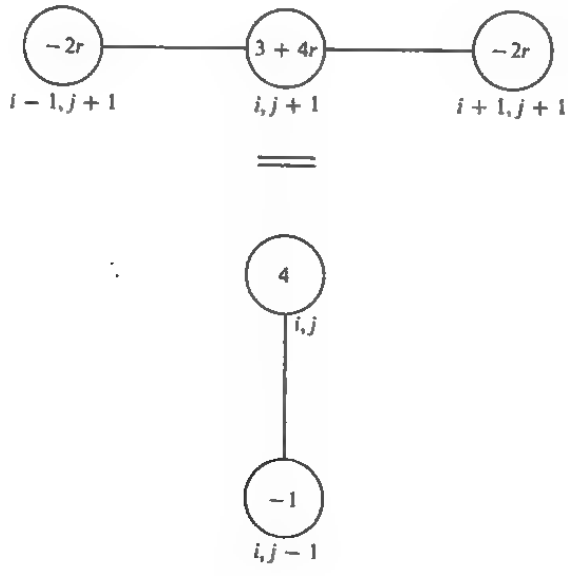
This scheme gives a tridiagonal set of equations, which can be solved by the recurrence form of Gauss elimination just as for the Crank–Nicolson scheme. The only differences between this scheme and the Crank–Nicolson are in the coefficients (which are again constants) and in the evaluation of the right-hand sides. Now we need more computation to evaluate the right-hand sides of the three-level scheme than in the Crank–Nicolson scheme. Therefore, on the grounds of efficiency, we would prefer the Crank–Nicolson scheme.

*Solution to SAQ 7*

Expanding the formula, we obtain

$$-2ru_{i+1,j+1} + (3 + 4r)u_{i,j+1} - 2ru_{i-1,j+1} = 4u_{i,j} - u_{i,j-1},$$

which has the following molecular diagram.




Once again a tridiagonal system of equations (with constant coefficients) must be solved. This scheme requires less arithmetic in evaluating the right-hand sides than does the Crank–Nicolson scheme.

*Solution to SAQ 8*

The Crank–Nicolson type formula for the equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 (U^m)}{\partial x^2}$$

is

$$\frac{u_{p,q+1} - u_{p,q}}{k} = \frac{1}{2h^2}(u^m_{p+1,q+1} - 2u^m_{p,q+1} + u^m_{p-1,q+1} + u^m_{p+1,q} - 2u^m_{p,q} + u^m_{p-1,q}).$$

The introduction of an error $E_{p,q}$ into $u_{p,q}$ given by

$$E_{p,q} = u^*_{p,q} - u_{p,q}$$

leads to the equation

$$\begin{aligned}(u^*_{p,q+1} - u_{p,q+1}) - (u^*_{p,q} - u_{p,q}) = \tfrac{1}{2}r[&u^{*m}_{p+1,q+1} - u^m_{p+1,q+1} - 2(u^{*m}_{p,q+1} - u^m_{p,q+1}) \\ &+ u^{*m}_{p-1,q+1} - u^m_{p-1,q+1} + u^{*m}_{p+1,q} - u^m_{p+1,q} \\ &- 2(u^{*m}_{p,q} - u^m_{p,q}) + u^{*m}_{p-1,q} - u^m_{p-1,q}]. \qquad (1)\end{aligned}$$

Assuming that $u$ varies slowly over a small region in comparison with the errors and that $E_{p,q}$ is small compared with $u_{p,q}$, we have

$$u_{p+1,q+1} \simeq u_{p,q+1} \simeq u_{p-1,q+1} \simeq u_{p+1,q} \simeq u_{p,q} \simeq u_{p-1,q}$$

and

$$u^{*m}_{p,q} - u^m_{p,q} \simeq mE_{p,q}u^{m-1}_{p,q}.$$

Equation (1) now becomes

$$E_{p,q+1} = E_{p,q} + \tfrac{1}{2}rmu^{m-1}_{p,q}(E_{p+1,q+1} - 2E_{p,q+1} + E_{p-1,q+1} + E_{p+1,q} - 2E_{p,q} + E_{p-1,q}).$$

Substituting the relation

$$E_{p,q} = e^{i\beta ph}\xi^q,$$

we have

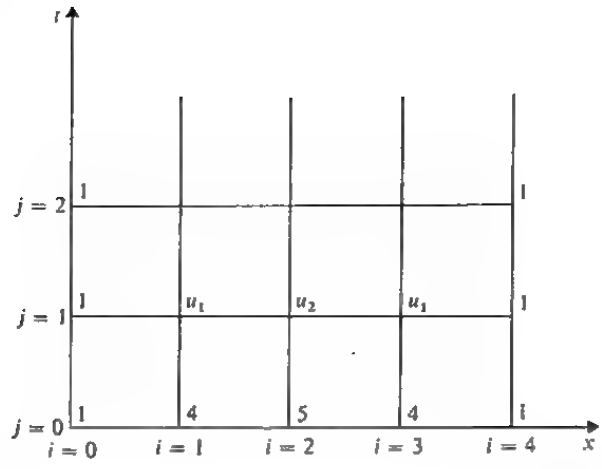
$$\xi = 1 + \tfrac{1}{2}rmu^{m-1}_{p,q}(e^{i\beta h}\xi - 2\xi + e^{-i\beta h}\xi + e^{i\beta h} - 2 + e^{-i\beta h})$$

on dividing by $e^{i\beta ph}\xi^q$. Putting $R = rmu^{m-1}_{p,q}$ in this equation and rearranging, we obtain

$$\xi\left(1 + 2R\sin^2\frac{\beta h}{2}\right) = 1 - 2R\sin^2\frac{\beta h}{2},$$

and clearly $|\xi| < 1$ for all values of $r$ provided $u^{m-1}_{i,j} \geqslant 0$. The method is therefore locally unconditionally stable.

*Solution to SAQ 9*




Let the pivotal values on the first time level be $u_1 = u_{1,1} = u_{3,1}$ and $u_2 = u_{2,1}$. The nonlinear equations which have to be solved are given by

$$f_1(u_1, u_2) \equiv u_2^2 - 2u_1^2 - 20u_1 + 75 = 0$$

and

$$f_2(u_1, u_2) \equiv 2u_1^2 - 2u_2^2 - 20u_2 + 82 = 0.$$

Differentiating we obtain

$$\frac{\partial f_1}{\partial u_1} = -4u_1 - 20, \quad \frac{\partial f_1}{\partial u_2} = 2u_2, \quad \frac{\partial f_2}{\partial u_1} = 4u_1, \quad \frac{\partial f_2}{\partial u_2} = -4u_2 - 20.$$

Starting the iteration with $(u_1^{(0)}, u_2^{(0)}) = (u_{1,0}, u_{2,0}) = (4, 5)$, we obtain

$$\begin{bmatrix} -36 & 10 \\ 16 & -40 \end{bmatrix} \begin{bmatrix} 0.656 \\ 1.162 \end{bmatrix} = \begin{bmatrix} -12 \\ -36 \end{bmatrix}$$

and so $(u_1^{(1)}, u_2^{(1)}) = (3.344, 3.838)$. The second iteration gives

$$\begin{bmatrix} -33.34 & 7.675 \\ 13.34 & -35.350 \end{bmatrix} \begin{bmatrix} -0.01234 \\ 0.05098 \end{bmatrix} = \begin{bmatrix} 0.8028 \\ -1.9667 \end{bmatrix}$$

from which $(u_1^{(2)}, u_2^{(2)}) = (3.356, 3.787)$. The third and final iteration is given by

$$\begin{bmatrix} -33.42 & 7.573 \\ 13.42 & -35.146 \end{bmatrix} \begin{bmatrix} 9.325 \times 10^{-3} \\ 1.346 \times 10^{-4} \end{bmatrix} = \begin{bmatrix} -0.3107 \\ 0.1205 \end{bmatrix}$$

and hence $(u_1^{(3)}, u_2^{(3)}) = (3.347, 3.786)$.

A convenient way of tackling this problem using the computer would be to use the library program $GEM201 which solves the matrix equations by the Gauss Elimination Method. The coefficients could be obtained from a simple program such as the following.

```
10  INPUT X1, X2
20  PRINT X2↑2−2*X1↑2−20*X1+75;−2*X2↑2+2*X1↑2−20*X2+82
30  PRINT −4 * X1 − 20; 2 * X2; 4 * X1; −4 * X2 − 20
40  STOP
50  END
```

*Solution to SAQ 10*

The local truncation error is given by

$$T_{i,j} = \frac{b_{i,j}}{2k}(U_{i,j+1} - U_{i,j-1})$$

$$-\frac{1}{3h^2}[\alpha^+_{i,j}(U_{i+1,j+1}-U_{i,j+1}+U_{i+1,j}-U_{i,j}+U_{i+1,j-1}-U_{i,j-1})$$

$$-\alpha^-_{i,j}(U_{i,j+1}-U_{i-1,j+1}+U_{i,j}-U_{i-1,j}+U_{i,j-1}-U_{i-1,j-1})]$$

where $b_{i,j}$ represents $b(U_{i,j})$. By Taylor's Theorem

$$\frac{1}{2k}(U_{i,j+1}-U_{i,j-1})=\frac{\partial U_{i,j}}{\partial t}+O(k),$$

$$\frac{1}{3h^2}[U_{i+1,j+1}-U_{i,j+1}+U_{i+1,j}-U_{i,j}+U_{i+1,j-1}-U_{i,j-1}]$$

$$=\left[\frac{1}{h}\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{h}{6}\frac{\partial^3}{\partial x^3}+O(h^2)\right](U_{i,j}+O(k^2))$$

and

$$\frac{1}{3h^2}[U_{i,j+1}-U_{i-1,j+1}+U_{i,j}-U_{i-1,j}+U_{i,j-1}-U_{i-1,j-1}]$$

$$=\left[\frac{1}{h}\frac{\partial}{\partial x}-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{h}{6}\frac{\partial^3}{\partial x^3}+O(h^2)\right](U_{i,j}+O(k^2)).$$

Now, by repeated use of Taylor's Theorem,

$$\alpha^+_{i,j}=a\left(\frac{U_{i,j}+U_{i+1,j}}{2}\right)$$

$$=a\left(U_{i,j}+\frac{h}{2}\frac{\partial U_{i,j}}{\partial x}+\frac{h^2}{4}\frac{\partial^2 U_{i,j}}{\partial x^2}+O(h^3)\right)$$

$$=a_{i,j}+\frac{h}{2}\frac{\partial U_{i,j}}{\partial x}\frac{da_{i,j}}{dU}+\frac{h^2}{4}\frac{\partial^2 U_{i,j}}{\partial x^2}-\frac{da_{i,j}}{dU}+\frac{h^2}{8}\left(\frac{dU_{i,j}}{\partial x}\right)^2\frac{d^2a_{i,j}}{dU^2}+O(h^3)$$

and

$$\alpha^-_{i,j}=a\left(\frac{U_{i,j}+U_{i-1,j}}{2}\right)$$

$$=a_{i,j}-\frac{h}{2}\frac{\partial U_{i,j}}{\partial x}\frac{da_{i,j}}{dU}+\frac{h^2}{4}\frac{\partial^2 U_{i,j}}{\partial x^2}\frac{da_{i,j}}{dU}+\frac{h^2}{8}\left(\frac{\partial U_{i,j}}{\partial x}\right)^2\frac{d^2a_{i,j}}{dU^2}+O(h^3),$$

where $a_{i,j}$ represents $a(U_{i,j})$. Hence

$$T_{i,j}=b_{i,j}\frac{\partial U_{i,j}}{\partial t}+O(k^2)-(\alpha^+_{i,j}+\alpha^-_{i,j})\left[\frac{1}{2}\frac{\partial^2 U_{i,j}}{\partial x^2}+O(h^2)+O(k^2)\right].$$

$$-(\alpha^+_{i,j}-\alpha^-_{i,j})\frac{1}{h}\left[\frac{\partial U_{i,j}}{\partial x}+O(h^2)+O(k^2)\right]$$

$$=b_{i,j}\frac{\partial U_{i,j}}{\partial t}-a_{i,j}\frac{\partial^2 U_{i,j}}{\partial x^2}-\left(\frac{\partial U_{i,j}}{\partial x}\right)^2\frac{da_{i,j}}{dU}+O(h^2)+O(k^2).$$

The differential equation gives

$$b(U)\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(a(U)\frac{\partial U}{\partial x}\right)$$

$$=\frac{\partial a}{\partial x}\frac{\partial U}{\partial x}+a(U)\frac{\partial^2 U}{\partial x^2}$$

$$=\left(\frac{\partial U}{\partial x}\right)^2\frac{da}{dU}+a(U)\frac{\partial^2 U}{\partial x^2},$$

and so

$$T_{i,j}=O(h^2)+O(k^2).$$

*Solution to SAQ 11*

The local truncation error $T_{i,j,n}$ is given by

$$T_{i,j,n} = \frac{U_{i,j,n+1} - U_{i,j,n}}{k} - \frac{U_{i+1,j,n} + U_{i-1,j,n} + U_{i,j+1,n} + U_{i,j-1,n} - 4U_{i,j,n}}{h^2}.$$

Taylor's Theorem gives

$$U_{i,j,n+1} - U_{i,j,n} = k\frac{\partial U_{i,j,n}}{\partial t} + \frac{k^2}{2}\frac{\partial^2 U_{i,j,n}}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 U_{i,j,n}}{\partial t^3} + \cdots,$$

$$U_{i+1,j,n} + U_{i-1,j,n} = 2U_{i,j,n} + h^2\frac{\partial^2 U_{i,j,n}}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 U_{i,j,n}}{\partial x^4} + \cdots$$

and

$$U_{i,j+1,n} + U_{i,j-1,n} = 2U_{i,j,n} + h^2\frac{\partial^2 U_{i,j,n}}{\partial y^2} + \frac{h^4}{12}\frac{\partial^4 U_{i,j,n}}{\partial y^4} + \cdots.$$

Therefore,

$$T_{i,j,n} = \frac{\partial U_{i,j,n}}{\partial t} - \frac{\partial^2 U_{i,j,n}}{\partial x^2} - \frac{\partial^2 U_{i,j,n}}{\partial y^2} + \frac{k}{2}\frac{\partial^2 U_{i,j,n}}{\partial t^2} - \frac{h^2}{12}\left[\frac{\partial^4 U_{i,j,n}}{\partial x^4} + \frac{\partial^4 U_{i,j,n}}{\partial y^4}\right] + \cdots.$$

The differential equation gives

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} = 0$$

and so

$$T_{i,j,n} = O(k) + O(h^2).$$

We see that

$$\lim T_{i,j,n} = 0 \text{ as } h, k \rightsquigarrow 0$$

and the given formula is consistent with the two-dimensional heat equation.

*Solution to SAQ 12*

(a) Suppose that we introduce an error at each point along a time level such that we perform the calculations with $\mathbf{u}^*$ replacing $\mathbf{u}$, and

$$E_{p,q,n} = u_{p,q,n} - u^*_{p,q,n}.$$

The error $E_{p,q,n}$ then satisfies the difference equation given by

$$E_{p,q,n+1} = E_{p,q,n} + r(E_{p+1,q,n} + E_{p-1,q,n} + E_{p,q+1,n} + E_{p,q-1,n} - 4E_{p,q,n}).$$

Putting $E_{p,q,n} = e^{i\beta_1 ph}e^{i\beta_2 qh}\zeta^n$, we obtain

$$\zeta = 1 + r(e^{i\beta_1 h} + e^{i\beta_2 h} + e^{-i\beta_1 h} + e^{-i\beta_2 h} - 4)$$

after dividing through by $e^{i\beta_1 ph}e^{i\beta_2 qh}\zeta^n$. Since

$$e^{i\beta h} + e^{-i\beta h} = 2\cos\beta h$$

$$= 2 - 4\sin^2\left(\frac{\beta h}{2}\right),$$

we obtain

$$\zeta = 1 - 4r\left[\sin^2\left(\frac{\beta_1 h}{2}\right) + \sin^2\left(\frac{\beta_2 h}{2}\right)\right].$$

Now, for stability $|\zeta| \leqslant 1$ and so we require

$$-1 \leqslant 1 - 4r\left[\sin^2\left(\frac{\beta_1 h}{2}\right) + \sin^2\left(\frac{\beta_2 h}{2}\right)\right] \leqslant 1.$$

The right-hand inequality is trivially satisfied for $r \geqslant 0$ whilst the left-hand inequality gives

$$r \leqslant \frac{1}{2\left[\sin^2 \frac{\beta_1 h}{2} + \sin^2 \frac{\beta_2 h}{2}\right]},$$

and since $0 \leqslant \sin^2 \frac{1}{2}\beta_1 h \leqslant 1$ and $0 \leqslant \sin^2 \frac{1}{2}\beta_2 h \leqslant 1$ we get the stability condition $r \leqslant \frac{1}{4}$.

(b) As the scheme is stable if $r \leqslant \frac{1}{4}$, and is consistent with the given differential equation, it is also convergent by Lax's Theorem. (Note that both the differential and difference equations are linear.)

*Solution to SAQ 13*

The finite-difference method gives

$$u_{i,j,n+1} - \frac{r}{2}(u_{i+1,j,n+1} + u_{i-1,j,n+1} + u_{i,j+1,n+1} + u_{i,j-1,n+1} - 4u_{i,j,n+1})$$

$$= \text{known quantities},$$

i.e.,

$$u_{i,j,n+1} - \frac{r}{2(1+2r)}(u_{i+1,j,n+1} + u_{i-1,j,n+1} + u_{i,j+1,n+1} + u_{i,j-1,n+1})$$

$$= \text{known quantities}.$$

The collection of all equations for $1 \leqslant i, j \leqslant M-1$ can be written as

$$A\mathbf{u} = \mathbf{b}. \tag{1}$$

This formula has precisely the structure of the five-point Laplace formula discussed in *Unit 11*; therefore the discussion about SOR applies provided that the associated Jacobi iteration converges. The natural ordering (Section 11.4.2 of *Unit 11*) leads to the following form for the matrix $A$ in Equation (1):

$$A = \theta \begin{bmatrix} B & I & & & & & \\ I & B & I & & & & \\ & I & B & I & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & I & B & I \\ & & & & & I & B \end{bmatrix}$$

where $A$ is square of order $(M-1)^2$, $\theta = -r/[2(1+2r)]$, $B$ is a square matrix of order $M-1$ given by

$$B = \begin{bmatrix} \theta^{-1} & 1 & & & & & \\ 1 & \theta^{-1} & 1 & & & & \\ & 1 & \theta^{-1} & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & \theta^{-1} & 1 \\ & & & & & 1 & \theta^{-1} \end{bmatrix}$$

and $I$ is the identity matrix of order $M-1$. The associated Jacobi iteration matrix $J$ is given by

$$J = I - A = -\theta \begin{bmatrix} D & I & & & & & \\ I & D & I & & & & \\ & I & D & I & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & I & D & I \\ & & & & & I & D \end{bmatrix}$$

where

$$D = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}$$

is of order $M-1$. By the result of the appendix to *Unit 11*, the eigenvalues of $J$ are given by

$$\alpha_{p,q} = \frac{r}{1+2r}\left[\cos\left(\frac{p\pi}{M}\right) + \cos\left(\frac{q\pi}{M}\right)\right] \qquad p, q = 1, 2, \ldots, M-1. \tag{2}$$

Since

$$-1 < \cos\frac{p\pi}{M}, \cos\frac{q\pi}{M} < 1 \qquad p, q = 1, 2, \ldots, M-1$$

we see that

$$\frac{-2r}{1+2r} < \alpha_{p,q} < \frac{2r}{1+2r}$$

and therefore $|\alpha| < 1$ and the Jacobi iteration is convergent. The SOR method is therefore convergent too. To find the optimum relaxation factor $\omega_{opt}$ we use the formula

$$\omega_{opt} = \frac{2}{1 + [1 - \rho(J)^2]^{\frac{1}{2}}}$$

as given in Section 11.4.3 of *Unit 11*, where $\rho(J)$ is the spectral radius of the Jacobi iteration matrix. Now, from Equation (2),

$$\rho(J) = \frac{2r}{1+2r}\cos\frac{\pi}{M}$$

giving

$$\omega_{opt} = \frac{2}{1 + \left[1 - \left(\frac{2r}{1+2r}\right)^2 \cos^2\frac{\pi}{M}\right]^{\frac{1}{2}}}.$$

# Unit 16 Blood Flow in Arteries

## Contents

## Set Books

G. D. Smith, *Numerical Solution of Partial Differential Equations* (Oxford, 1971).

H. F. Weinberger, *A First Course in Partial Differential Equations* (Xerox, 1965).

It is essential to have these books; the course is based on them and will not make sense without them. They are referred to in the text as *S* and *W* respectively.

*Unit 16* is not based on either set book.

## Conventions

Before working through this text make sure you have read *A Guide to the Course: Partial Differential Equations of Applied Mathematics.* Reference to Open University courses in mathematics take the form:

*Unit M100 13, Integration II* for the Mathematics Foundation Course,
*Unit M201 23, The Wave Equation* for the Linear Mathematics Course.

## Bibliography

J. R. Womersley, *An Elastic Tube Theory of Pulse Transmission and Oscillatory Flow in Mammalian Arteries* (Wright Aircraft Development Centre Technical Report TR 56–614, January 1957).

D. A. McDonald, *Blood Flow in Arteries* (Edward Arnold, 1960).

The mathematical theory used in this unit is taken from these two references. The comparison with experiments and most of the background material is taken from the Ph.D. Thesis of Peter Hutton (University of Birmingham, 1973).

M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, 1965).

This book contains formulas, graphs and tables of the Bessel functions which are used in this unit.

## Acknowledgement

We gratefully acknowledge the permission and assistance given by Dr. D. H. Tompsett and The Royal College of Surgeons of England for allowing us to photograph the corrosion cast showing the cerebral arteries of the human brain used in the cover design.

# 16.0 INTRODUCTION

It is unnecessary to stress the importance of understanding the flow of blood in the circulatory system. This system is extremely complex and it is unlikely that a satisfactory mathematical theory of the whole of it will ever be obtained; indeed, at present we are some way from obtaining an accurate mathematical model of arterial flow.

This unit describes some of the work done with a view to understanding the formation of thick plaques in the arterial wall which partially block the artery. This disease, called atherosclerosis, is the chief cause of male deaths in Western societies (*New Scientist*, 12 July 1973, p. 65). The reason for the formation of these plaques is not understood, but sometimes the artery becomes sufficiently restricted to warrant remedial surgery, which may consist of the insertion of a by-pass; this is either an artificial artery or a segment of the patient's vein. Unfortunately it is sometimes found that both the parent artery and the graft tend to block again at the down stream end.

There is evidence that, in the formation of plaques, the chemical composition of the blood is an important factor, but now physical properties of the flow are also considered important. At present it seems that a combination of chemical and physical effects are important in the formation of plaques. The work described here is an attempt at understanding some of the physical properties of blood flow.

In order to obtain some understanding of flow in an artery there are three possible approaches. Firstly, experiments can be performed on animals; secondly, we can make models in the laboratory; finally, we can make mathematical models and experiment with them. In practice the first option is not entirely satisfactory for detailed observations, because of the small diameter of the arteries in small animals and because of the difficulty of doing accurate experiments *in situ*, although such experiments are necessary before the other two methods can be used.* The mathematical models are usually so idealized that, without further evidence, their results cannot be relied upon to give accurate quantitative results. One of the problems is the very difficult task of gauging accurately the errors produced by making the approximations. However, the theory can give an insight into qualitative relations and into the likely important parameters. Physical models can be made of systems too complex to be modelled mathematically, and can give some idea of the significance of the approximations made in the mathematical model; but these models are subject to their own errors. In practice a combination of all three methods is necessary to make progress.

In this unit we shall consider an elementary mathematical model, but where possible we shall compare our results with experiments. Before considering blood flow in more detail we consider some general properties of *viscous flow* (Section 16.1), and in Section 16.2 we shall derive some equations pertaining to flow in cylindrical tubes. In Section 16.3 we consider in more detail the approximations made; and in Section 16.4 we apply the work of the preceding sections to blood flow. In Section 16.5 we compare our results with experiments.

Note that in this text we do not restrict ourselves to the customary SI units; this is because most of the literature on this subject uses the cgs system. The symbol $\mu$ is used for two distinct purposes; the first is to denote the coefficient of *viscosity* of a fluid, and the second is to denote one *micron* ($=10^{-6}$ m). There should be no confusion between these two uses, both of which follow convention. Finally, following the usual practice, pressures are measured in *millimetres of mercury* (mm Hg); since the density of mercury is 13.6 gm/cc, the pressure needed to support 1 mm Hg is 133.4 $Nm^{-2}$ or 0.019 lb per sq. in.

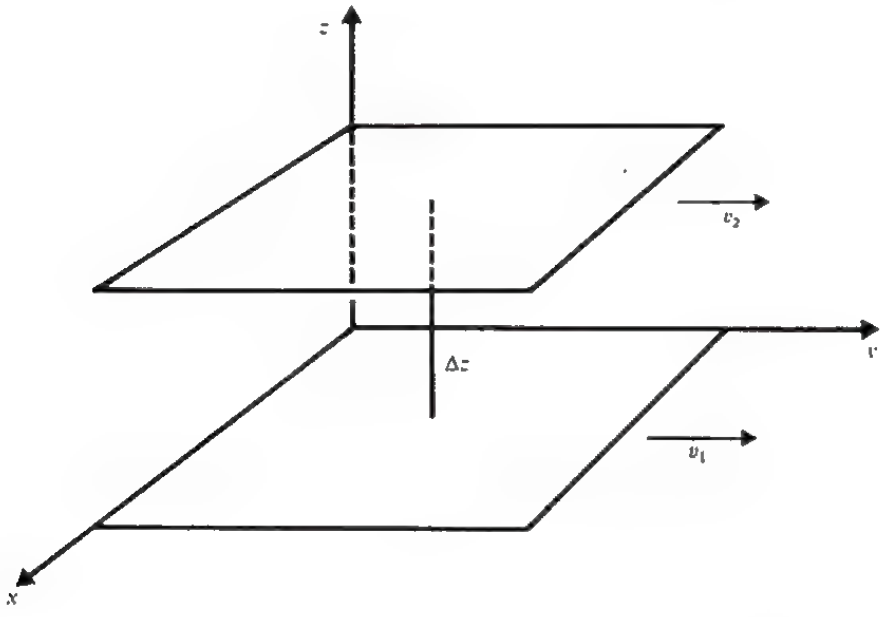
* A rat will be mailed in the next package.

## 16.1 LAMINAR FLOW IN VISCOUS FLUIDS

Throughout this unit we shall be concerned only with *laminar* flow, that is, flow in which the fluid moves as if it were in layers with different velocities. Not all flow is laminar, as can be seen by observing the passage of an object through water; if the velocity is small the fluid flow will be laminar, but if it is large enough eddies will form and the motion will be *turbulent*. In the case of turbulent flow the equation of motion is nonlinear, so a mathematical description of the flow is very difficult. In general, however, for the case that we are considering in this unit the flow is laminar.

All liquids are viscous, but in some cases the viscosity may be neglected; whether it can be neglected depends not just upon the liquid but also on the circumstances of the flow. In blood flow conditions are such that viscosity cannot be ignored.

If portions of a liquid are caused to move relatively to one another, the differential motion gradually subsides unless sustained by external forces; conversely, if a portion of a mass of liquid is kept moving, the motion gradually communicates itself to the rest of the liquid. We require a mathematical description of these effects, which are evident from immediate observation.




Newton was the first to formulate a hypothesis regarding viscous forces. His hypothesis amounts to the following: if two parallel plane laminae of area $A$ move in the same direction with different constant velocities $v_1$ and $v_2$, parallel to their planes, the force required to maintain this differential motion is

$$\frac{\mu A}{\Delta z}(v_1 - v_2),$$

where $\Delta z$ is the (perpendicular) distance between the laminae. Since the velocity of the fluid changes continuously we may replace the differences by a derivative, so that

$$F = \mu A \lim_{\Delta z \to 0} \frac{v_1 - v_2}{\Delta z}$$

$$= \mu A \frac{\partial v}{\partial z}.$$

The constant $\mu$ is called the coefficient of **viscosity**; it has the dimensions $ML^{-1}T^{-1}$ and in the cgs system the unit of viscosity is called a **poise**, in honour of Poiseuille. In practice, the convenient unit is one hundredth of a poise or a **centipoise** (cp); the viscosity of water is about 1 cp.

Newton did not pursue the subject very far (although he did consider the case of flow between two concentric cylinders), nor was the problem studied again for more

than a century. Nevertheless this first contribution is commemorated in the use of the term *Newtonian fluid* for simple viscous fluids. A Newtonian fluid is one whose viscosity does not depend on the velocity gradient.

The eighteenth century produced many great mathematicians, of whom Leonard Euler and Daniel Bernoulli devoted much thought to hydrodynamics, but their work was confined to ideal fluids, that is, fluids without viscosity. At the end of the century Coulomb studied the damping of the oscillations of a disc suspended in liquids of different viscosities and made the important observation that the smoothness or roughness of the surface of the disc did not greatly influence the drag of the liquid.

The first work on flow in cylindrical tubes appears to be that of Girard in 1813 using brass tubes of 2 to 3 mm in diameter. He obtained the relationship

$$Q \propto \frac{PR^3}{L},$$

where $Q$ is the volume flow per unit time, $R$ is the radius of the tube and $P$ the pressure drop along the length $L$ of the tube. Thus he observed that the flow varied directly with the pressure and inversely with the length but thought that it varied with the cube of the radius. Ten years later Navier derived theoretical equations for flow of viscous liquids in cylindrical tubes but obtained the incorrect result, apparently confirmed by Girard's experimental results, that the flow was proportional to the cube of the radius.

The first published experimental work indicating that the flow is proportional to the fourth power of the radius was that of Hagen in 1839. He used brass tubes of a size similar to those of Girard and his results were not very accurate. The exponent of the radius derived from his results was actually 4.12 and he assumed that the real value must be 4.0. Poiseuille published his first results in 1842, and a detailed paper in 1846. On account of this paper, laminar flow in tubes is often referred to as *Poiseuille flow*.

Poiseuille spent much time investigating the hydrodynamics of the capillary circulation. He may be regarded as fortunate in two respects. In the first place, although he had a training in physics, he was also a physician who wanted to apply the results of his investigation to an understanding of the blood circulation. Hence he worked with glass tubes of capillary size (between 0.14 and 0.03 mm) whereas his predecessors had been engineers who worked with much larger pipes. The use of smaller tubes enables laminar flow to be maintained more easily and also accurate measurement is facilitated greatly. In the second place he was deflected from his original intention of using blood as a test liquid because no satisfactory way of rendering it incoagulable was known and he was compelled to confine his investigation to water. Blood flowing in capillaries shows anomalous viscous properties that would have introduced complications in these pioneer studies.

Poiseuille's results are expressed by the formula

$$Q = \frac{KPR^4}{L} \tag{1}$$

where $Q$, $P$, $R$ and $L$ are defined above. The value of the constant $K$ was determined under various conditions and shown to fall with increasing temperature. This constant is clearly a measure of the viscosity, but by purely experimental work it is not possible to relate it exactly to the coefficient of viscosity defined earlier.

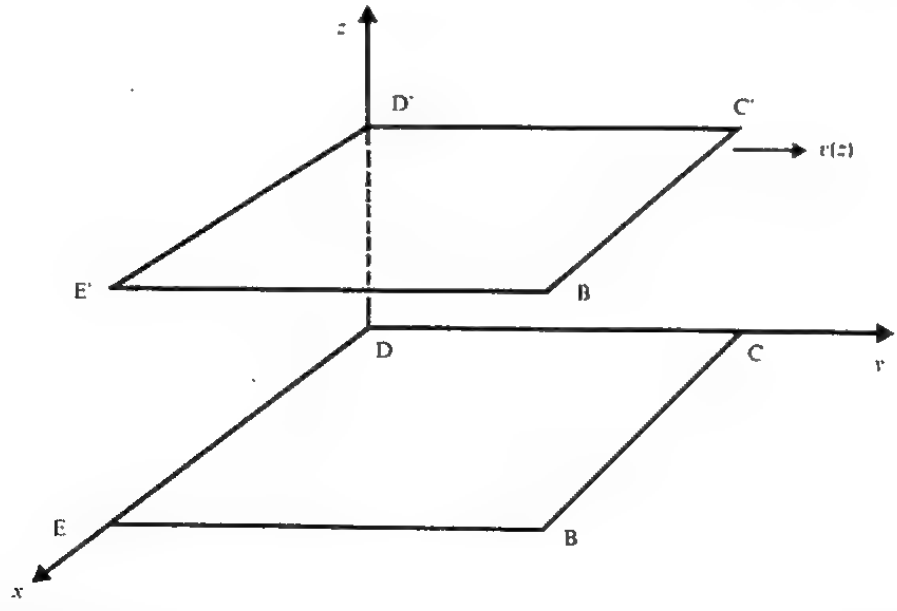
A theoretical derivation of Equation (1) may be obtained from the *Navier-Stokes equation*. The early work of Navier on the equation of motion for viscous fluids was amplified and corrected by Stokes in the 1840s; the Navier-Stokes equation is the mathematical formulation of the problem. Stokes, however, did not consider the particular case of flow in a tube; the solution to this problem was obtained independently by Wiedemann in 1856 and Hagenbach in 1860 who showed that

$$Q = \frac{(P_1 - P_2)}{L} \frac{\pi R^4}{8\mu} \tag{2}$$

where $\mu$ is the viscosity and $P_1$ and $P_2$ the pressures at the ends of a tube of length $L$. That this relatively trivial solution to the Navier-Stokes equation took so long to find is probably a reflection on the fact that few people were interested in the problem.

We now consider viscosity in more detail. Viscosity is the outcome of two distinct causes: the motion of the molecules, and the action of the intermolecular forces. The former is operative mainly in gases, where the molecules are free to traverse relatively long distances (of the order of $10^{-7}$ m at ordinary pressures, this being very much larger than the radius of a molecule, typically $10^{-10}$ m) between collisions with one another. Suppose that we have a fluid in which two adjacent layers are moving at different velocities; the average velocities of the molecules in these two layers will differ by the same amount. This means that the molecules arriving in the faster moving layer from the other will have, on the average, a smaller velocity than those already present, and conversely, the molecules arriving in the more slowly moving layer from the other will have, on the average, a greater velocity than those already present. This tendency to remove inequalities in the macroscopic velocity distribution is the salient characteristic of viscosity; in the absence of externally imposed stresses to maintain the flow, it would ultimately bring the entire fluid to a state of uniform motion or rest.

In liquids, however, the molecular mechanism which tends to dissipate the motion of the fluid is essentially different from that already described as operative in gases. Being surrounded by other molecules, the individual molecule is not able to travel freely from one layer to another, and actually follows a very erratic path which is not well adapted to the transport of momentum. There is, however, another very much more effective method of exchange of momentum, which depends on the continuous action of the intermolecular forces. Perhaps the easiest method of visualizing the nature of the process involved is to imagine that the molecules are connected by elastic strings which simulate the action of the intermolecular forces. It can then be seen that, if two adjacent layers in the fluid move with different velocities, each will tend to drag the other in such a manner as to dissipate the relative motion, in the absence of sustaining external stresses. This method of exchange of momentum is most effective when the molecules are close together, which explains why it is rather unimportant in gases, yet the dominant process in liquids. An immediate observation is that viscous forces between the layers of fluid are proportional to the area of contact.




Consider a fluid as shown in the figure above such that the fluid is uniform and such that each plane parallel to the $xy$-plane is moving at a constant velocity $v(z)$ in the $y$-direction. If the velocity is independent of $z$ the fluid is moving bodily, as though solid, and from what we have said there will be no viscous forces, except at the boundaries, which we shall ignore for the present. Now suppose that $v(z)$ varies with $z$; we seek the magnitude of the force $F$ on the plane BCDE due to the relative motion

of the plane B'C'D'E', each of which has area $A$. As we have seen, the force is proportional to the area; so we consider the force per unit area, $F/A$. We expect this to depend upon the velocity gradient $\partial v/\partial z$, so that

$$\frac{F}{A} = f\left(\frac{\partial v}{\partial z}\right)$$

$$= f(0) + \left(\frac{\partial v}{\partial z}\right) f'(0) + \left(\frac{\partial v}{\partial z}\right)^2 \frac{f''(0)}{2} + \cdots$$

where the last expression follows from Taylor's Theorem. Since $F = 0$ when $\partial v/\partial z = 0$, we have $f(0) = 0$. Thus, to first order in the velocity gradient,

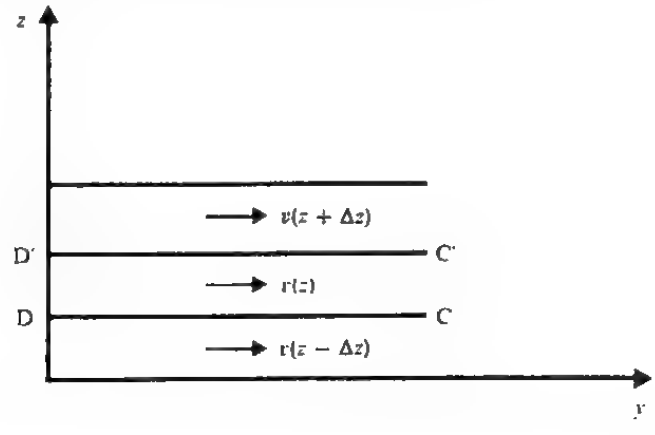
$$\frac{F}{A} = \mu \frac{\partial v}{\partial z}, \tag{3}$$

which is the result that Newton assumed. This expression gives the force per unit area on a plane surface parallel to the fluid flow.

This method of establishing a law about viscous forces may appear artificial, but in fact is typical of methods used for making mathematical models. First we isolate the significant variables—in our case we used a physical argument to suggest that viscous forces depend upon velocity gradients; this is the hardest part of the exercise and usually requires considerable insight and experimentation. Next a law relating these variables is postulated—in this instance the linear relation given by Equation (3). The law can be tested experimentally, or it can be derived from a more general theory. In the former case the arbitrary constants introduced, here $\mu$, will be determined as a number; in the latter case they will be determined in terms of more fundamental constants. Viscosity is an interesting example—for gases the constant $\mu$ can be derived from the Kinetic Theory of gases, but for liquids no satisfactory basic theory exists. Experimentally, the assumption that viscous forces depend linearly on the velocity gradient is often good even for quite large velocity gradients. However, blood is one of many fluids for which this is not sufficiently accurate, as we shall see in Section 16.3.

In order to determine the motion of a fluid element we require the force per unit volume acting on an element of the fluid.

Consider a fluid which is moving in the $y$-direction with a velocity which depends upon $z$, as in the previous figure. We suppose that the boundaries of the fluid are so far away from the region of interest that they may be ignored. We split the fluid into a series of horizontal layers each of height $\Delta z$, so that each moves as a rigid body with slipping occurring only at its boundaries, rather like a pack of cards. We suppose that the velocity of each layer is the velocity of the actual fluid at its centre, as in the following figure which shows a cross-section of the motion.




The velocity differences across the boundary planes CD and C'D' are then

$$\Delta v_{CD} = v(z) - v(z - \Delta z)$$
$$\Delta v_{C'D'} = v(z + \Delta z) - v(z) \tag{4}$$

and the velocity gradient is $\Delta v/\Delta z$, so that the viscous forces in the $y$-direction acting on a slab, of area $A$, between CD and C′D′ are

$$F_{CD} = -\mu A \frac{\Delta v_{CD}}{\Delta z}$$

and

$$F_{C'D'} = \mu A \frac{\Delta v_{C'D'}}{\Delta z}.$$

The force per unit volume in the $y$-direction acting on the layer between CD and C′D′ is then

$$F = \frac{F_{CD} + F_{C'D'}}{A\Delta z} = \frac{\mu}{\Delta z} \frac{\Delta v_{C'D'} - \Delta v_{CD}}{\Delta z}. \tag{5}$$

Using Taylor's Theorem in Equations (4) and inserting the result into (5) we obtain

$$F = \frac{\mu}{(\Delta z)^2}\left[(\Delta z)^2 \frac{\partial^2 v}{\partial z^2} + O(\Delta z)^4\right],$$

so that, in the limit as $\Delta z \to 0$, the viscous force per unit volume in the $y$-direction is

$$F = \mu \frac{\partial^2 v}{\partial z^2}. \tag{6}$$

We must now consider the boundary conditions associated with the motion. There are always forces of molecular attraction between a viscous fluid and the surface of a solid body, and these forces have the result that the layer of fluid immediately adjacent to the surface adheres to it, and is brought to rest. Accordingly, the boundary conditions for the motion of a viscous fluid require that the fluid velocity should vanish at fixed solid surfaces. In the general case of a moving surface, the velocity must be equal to the velocity of the surface.

The derivation of the Navier-Stokes equation for a general fluid is beyond the scope of this course. However, we are interested primarily in incompressible flow along cylindrical tubes, in which case the equations are relatively simple to derive.

Newton's Second Law of Motion gives us (see Appendix to *Unit 14, Bessel Functions*), for a fluid of density $\rho$,

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathbf{grad})\mathbf{v}\right) = -\mathbf{grad}\, p + \mathbf{F}_0,$$

where the operator $\mathbf{v} \cdot \mathbf{grad}$ acts on each coordinate of $\mathbf{v}$ and is given in Cartesian coordinates by

$$\mathbf{v} \cdot \mathbf{grad} = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z},$$

$\mathbf{F}_0$ comprises the body forces per unit volume (e.g. gravity, electromagnetic forces, and now the viscous forces), and $p$ denotes the pressure.

*SAQ 1*

Consider fluid flow between two fixed parallel planes, as in the preceding text, where the velocity $\mathbf{v}$ is in the $y$-direction and depends on $z$ and $t$ only. Show that $v$ satisfies the equation

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial z^2} + F,$$

where $F$ denotes the external force acting in the $y$-direction, $\rho$ denotes the density and $p$ denotes the pressure.

(Solution on p. 29.)

*SAQ 2*

(a) A viscous fluid at rest is bounded by the $yz$-plane and fills the region $x > 0$, with constant pressure and no external forces. If the boundary is now moved in the $y$-direction with given velocity $u(t)$ show that the equation of motion is the diffusion equation

$$\frac{\partial v}{\partial t}(x, t) = \nu \frac{\partial^2 v}{\partial x^2}(x, t) \qquad x > 0,$$

where $\nu = \mu/\rho$, and that $v(0, t) = u(t)$.

(b) If $u(t) = A \cos \omega t$, determine the velocity of the fluid, and find the force per unit area necessary to keep the plane moving in this manner.

(Solution on p. 29.)

The quantity $\nu = \mu/\rho$, which will appear frequently in this unit, is called the **kinematic viscosity**.

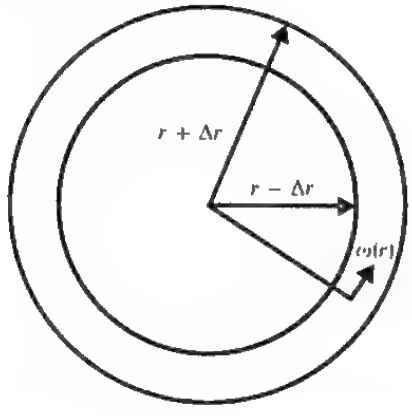
# 16.2 VISCOUS FLOW IN CYLINDRICAL TUBES

## 16.2.0 Introduction

Since we are interested in flow along tubes, we devote this section to deriving the equation of motion of a fluid moving in a cylindrical tube in two special cases. First, in order to illustrate the derivation of the equation of motion, we consider the motion of a fluid rotating about the axis of the cylinder. Next, we look at the more relevant problem of fluid moving parallel to the axis of the cylinder, and obtain a result we shall require in our investigation of blood flow.

## 16.2.1 Rotational Flow About an Axis

We consider radial flow in a cylinder, i.e. we suppose that the velocity is in the tangential direction only, and that all variables depend only on the radius $r$ and the time $t$. To obtain the forces acting on a fluid element, we split the fluid into a series of concentric rings of equal width $2\Delta r$, each supposed rigid and rotating with angular velocity equal to the actual angular velocity of the fluid at the middle radius of the ring. (The **angular velocity** is the rate of rotation, so that, if $\omega$ denotes the angular velocity of the ring, the tangential velocity of a particle of fluid at radius $r$ is given by $r\omega$.)




Let us consider the ring of inner radius $r - \Delta r$ and outer radius $r + \Delta r$, as shown in the figure above. To determine the viscous forces acting on this ring it is necessary to find the rate at which the two adjacent rings are sliding over it. Let $v$ denote the velocity of the fluid; then the angular velocity of the ring is given by

$$\omega(r, t) = \frac{1}{r} v(r, t),$$

and the angular velocity of the outer adjacent ring is given by

$$\omega(r + 2\Delta r, t) = \frac{v(r + 2\Delta r, t)}{r + 2\Delta r}.$$

By Taylor's Theorem,

$$v(r + 2\Delta r, t) = v(r, t) + 2\Delta r \frac{\partial v}{\partial r}(r, t) + 2(\Delta r)^2 \frac{\partial^2 v}{\partial r^2}(r, t) + O(\Delta r)^3$$

and

$$\left(1 + \frac{2\Delta r}{r}\right)^{-1} = 1 - \frac{2}{r}\Delta r + \frac{4}{r^2}(\Delta r)^2 + O(\Delta r)^3.$$

Hence

$$\omega(r + 2\Delta r, t) = \frac{1}{r}[v(r, t) + B(r, t)2\Delta r + A(r, t)2(\Delta r)^2 + O(\Delta r)^3],$$

where

$$A = \frac{\partial^2 v}{\partial r^2} - \frac{2}{r}\frac{\partial v}{\partial r} + \frac{2v}{r^2},$$

$$B = \frac{\partial v}{\partial r} - \frac{v}{r}.$$

The angular velocity difference at the boundary is now

$$\Delta\omega = \omega(r + 2\Delta r, t) - \omega(r, t) = \frac{2\Delta r}{r}[B(r, t) + A(r, t)\Delta r + O(\Delta r)^2]$$

and the relative velocity at the boundary is given by

$$\Delta v = (r + \Delta r)\Delta\omega.$$

The velocity gradient at the outer surface is then $\Delta v/(2\Delta r)$. We consider the viscous force acting between a segment of the ring shown in the previous figure and the adjacent outer ring. Let the angle subtended by the segment at the centre of the ring be $\Delta\theta$; then the viscous force is

$$\text{area} \times \mu \times \text{velocity gradient} = l(r + \Delta r)\Delta\theta\mu\frac{\Delta v}{2\Delta r},$$

where $l$ is the length of the cylinder.

Viscous forces act on the ring, in the tangential direction, at all points around its circumference. Because of this, these forces may not be added directly: it is necessary to add their turning effects about the axis of the cylinder. Formally, we define the **moment** about P of a force **F** acting through Q as $\mathbf{r} \times \mathbf{F}$, where $\mathbf{r} = \overrightarrow{PQ}$.* In our case we see that the magnitude of the moment of the force on the segment is

$$l(r + \Delta r)^2\Delta\theta\mu\frac{\Delta v}{2\Delta r},$$

so that the viscous forces on the outer face of the cylindrical shell exert a total moment

$$M_1 \simeq \frac{2\pi\mu l(r + \Delta r)^3}{r}[B(r, t) + A(r, t)\Delta r] \tag{1}$$

about the axis.

The moment of the viscous forces due to the motion of the inner ring may be obtained in the same way, but it is easier merely to change the sign of $\Delta r$ in Equation (1), being careful to note that, on this surface, the moment will be acting in the direction opposite to that of $M_1$. The moment is therefore

$$M_2 \simeq -\frac{2\pi\mu l(r - \Delta r)^3}{r}[B(r, t) - A(r, t)\Delta r].$$

Since the volume of the ring is $4\pi rl\Delta r$, the moment per unit volume is given by

$$\frac{M_1 + M_2}{4\pi rl\Delta r} \simeq \frac{\mu}{2r^2}\left\{B\frac{(r + \Delta r)^3 - (r - \Delta r)^3}{\Delta r} + A[(r + \Delta r)^3 + (r - \Delta r)^3]\right\}$$

$$\simeq \mu r\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right] \qquad \text{neglecting terms of order } (\Delta r)^2.$$

But the **angular momentum** of the ring is $rv\rho \times$ volume, where $\rho$ is the density of the fluid, so that the rate of change of angular momentum per unit volume is $r\rho\partial v/\partial t$, and this is equal to the moment of the forces acting per unit volume†. Thus we have

$$r\rho\frac{\partial v}{\partial t} = \mu r\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right). \tag{2}$$

* For a discussion of moments, see *Unit MST 282 1, Some Basic Tools.*

† If $\mathbf{l} = m\mathbf{r} \times \dot{\mathbf{r}}$ is the angular momentum, $\dot{\mathbf{l}} = m\ddot{\mathbf{r}} \times \dot{\mathbf{r}} + m\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times (m\ddot{\mathbf{r}})$ since $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = \mathbf{0}$. Using Newton's Second Law of Motion, $\mathbf{F} = m\ddot{\mathbf{r}}$, it follows that $\dot{\mathbf{l}} = \mathbf{r} \times \mathbf{F} =$ moment of the forces.

*SAQ 3*

(a) Determine the velocity of a fluid between two concentric cylinders of radii $a, b$ $(b > a)$ when the angular velocity of the outer cylinder is $\Omega(t)$ and the inner cylinder is maintained at rest, in the two cases:

(i) $\Omega(t) =$ constant,

(ii) $\Omega(t) = A \cos \omega t$, where $\omega$ is a constant.

NOTE: In solving (ii) you should obtain the equation

$$z^2 \frac{d^2V}{dz^2} + z\frac{dV}{dz} + (z^2 - m^2)V = 0$$

with $m = 1$. This is Bessel's equation, which has a solution $J_m(z)$ introduced in *Unit 14*. However, being a second order differential equation, there are two linearly independent solutions, and another solution is denoted by $Y_m(z)$. This solution was not discussed in *Unit 14*, since there we were looking for a solution which was bounded at the origin, whereas $Y_m$ is unbounded at the origin. In this SAQ the behaviour at the origin is irrelevant and both solutions are needed; you will not require any properties of $Y_m$.

(b) Determine the moment per unit length on the inner cylinder in case (i).

(c) Show that the kinetic energy of the fluid per unit length of the cylinder is

$$E = \pi\rho \int_a^b rv^2 \, dr$$

where $v$ is the tangential velocity of the fluid and $\rho$ is its density. Show also that the rate of change of the kinetic energy is

$$\frac{dE}{dt} = 2\pi\rho \int_a^b rv\frac{\partial v}{\partial t} \, dr.$$

(d) Using the equation of motion, Equation (2), and integrating by parts, show that

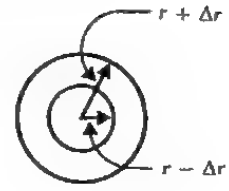
$$\frac{dE}{dt} = 2\pi\mu\left[rv\frac{\partial v}{\partial r}\right]_a^b - 2\pi\mu\int_a^b \left[r\left(\frac{\partial v}{\partial r}\right)^2 + \frac{v^2}{r}\right] dr.$$

Interpret each of these terms physically.

(Solution on p. 30.)

## 16.2.2 Flow Parallel to an Axis

We now consider the case of fluid flowing parallel to the axis of a cylinder.


Consider a tube with axis along the $x$-axis, and in this tube a cylindrical shell of length $l$, inner radius $r - \Delta r$ and outer radius $r + \Delta r$, centred upon the axis of the tube. Suppose that the fluid is flowing parallel to the axis and that its velocity varies only in the radial direction: let this velocity be $w(r, t)$.

*SAQ 4*

Show that the viscous forces on the inner and outer surfaces of the shell are

$$2\pi\mu l\left[r\frac{\partial w}{\partial r}\right]_{r=r\pm\Delta r}$$

in the positive $x$-direction. Hence show that the total force on this fluid element is

$$-4\pi r\Delta r[p(x+l)-p(x)]+2\pi\mu l\left[r\frac{\partial w}{\partial r}\right]_{r-\Delta r}^{r+\Delta r},$$

where $p$ denotes the pressure, which may vary with time (although this dependence is not explicitly noted), and the shell lies between $x$ and $x + l$.

(Solution on p. 32.)

Since the motion is parallel to the axis and the fluid is incompressible, we can use Newton's Second Law of Motion directly. The mass of the cylinder is $4\pi rl\rho\Delta r$, where $\rho$ is the (constant) density of the fluid, and its acceleration is $\partial w/\partial t$.

*SAQ 5*

Write down the equation of motion for the cylinder, and, by taking the limits as $l \rightsquigarrow 0$ and $\Delta r \rightsquigarrow 0$, derive the equation

$$\frac{\partial w}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right). \tag{3}$$

(Solution on p. 32.)

A consequence of Equation (3) is that, if the pressure gradient is zero (constant pressure), the only possible solution in the steady state (all variables independent of time) is $w(r, t) = 0$, since the velocity is zero at the walls. Thus in order to obtain a steady flow the pressure must drop in the direction of flow, a balance between the pressure gradient and the viscous drag on the walls being achieved. We shall suppose that Equation (3) describes blood flow, and in Section 16.4 it will be solved with the appropriate pressure gradient. In the next section (16.3) we shall consider, in slightly more detail, the assumptions that are made and how they are likely to affect our results. However, before continuing, we ask you to derive a result which we shall require in Section 16.4.

*SAQ 6*

Determine the velocity distribution for steady flow along a cylinder of radius $R$ with a constant pressure gradient $-\Delta p$.

(Solution on p.32.)

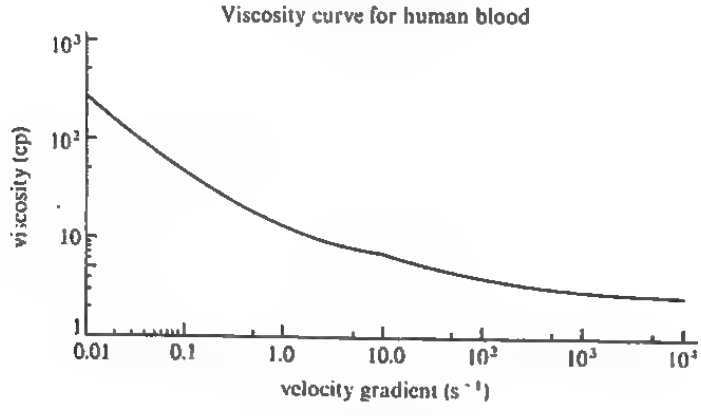
## 16.3 APPLICABILITY OF THE EQUATION OF MOTION TO BLOOD FLOW

Before going further and applying the equation of motion to blood flow, we shall consider whether our assumptions are valid in this case, and how we can test whether the solutions to our equations predict accurately what really happens. First we consider the assumptions made.

We are assuming that blood is incompressible, homogeneous, and that its viscosity does not vary with the velocity gradient. The first of these assumptions seems reasonable and we shall not comment on it any further; the second and third are rather dubious. The arterial wall is approximated by a straight, rigid tube, long enough for the disturbance of the fluid entering it to be neglected; all of these assumptions could lead to serious discrepancies.

Let us now consider blood in more detail. There are two main constituents of blood: there is the plasma, which is a transparent, very pale yellow liquid with a density close to that of water (1.03 gm/cc), and there are the red blood cells, whose major function is to transport oxygen. The red cells are roughly disc-shaped with a diameter and thickness of about $8\mu$ and $2\mu$ respectively; they occupy, on average, 40–45% by volume of blood. The radius of a cell is roughly that of a small capillary; consequently our fluid model could not work for capillary flow. Since the radius of an artery is typically $10^{-2}$ m ($10^{4}\ \mu$) we should expect that there are circumstances for which the red cells may be neglected; it is known, however, that under certain circumstances they can affect the flow significantly although, in general, their effect on the flow is not understood.

One effect of the red cells is on the viscosity, which varies with the velocity gradient; the exact dependence is not known and experimental results often differ by a factor of 5 or more. However, in the following figure we show the general behaviour of the viscosity with the velocity gradient.




To obtain a full understanding of blood flow it is obviously necessary to take into account the motion of the red cells. This is difficult and so far has not been done. However, in a first approximation it is convenient to ignore them and to proceed assuming that the viscosity is constant. There is some experimental evidence suggesting that, in the situation that we consider, this is not such a bad approximation.

Neglect of the red cells and their complicated effects implies that a pressure wave travelling in the fluid moves without change of shape; this is a consequence of the equations of motion being linear. But, we should expect the red cells to interact with a pressure wave by absorption and reflection, thereby changing its shape; the amount by which its shape is changed will be some indication of whether or not our assumption is justified. Of course, this distortion will depend upon the frequency of the wave; at low frequencies, such that the wave lengths are large in comparison with the cell size, the cells will not be distorted by the wave since the pressure will be the same all over a cell, which will therefore not interfere seriously with the wave. At high frequencies, however, when the wave length is approximately the size of a cell, different

parts of the cell experience different forces, so that the pressure wave will both move and deform the cell, and this motion will interfere with the wave.

To test this idea, an experiment measuring the shape of a pressure wave at frequencies comparable with that of a heart beat (about 1 cycle/sec) has been performed; the wave velocity was about 1500 m/sec giving a wave length of about 1500 m, which is rather larger than the size of a cell. At these frequencies no measurable effect was found, so that we can conclude tentatively that our assumption is justified.

Now we consider the artery. In reality this is a flexible tube with a structure sufficiently complicated to ensure that its elastic properties are nonlinear; this was observed as early as 1808. However, we shall have to ignore these problems; a mathematical model has been developed assuming that the elasticity is linear and that the tube is straight, and this is the next stage following the work described in this unit. It transpires that the effects of elasticity are small, but not small enough to be ignored completely; the theory presented here should be regarded as the necessary first stage of a more complicated theory.

In our mathematical analysis we assume the tube to be long and to have no branches. These are poor assumptions; branches in particular have a significant effect, and indeed, as mentioned in the Introduction, the behaviour there is of prime interest. Unfortunately, at present only experiments can provide us with the information in these circumstances, although computer solutions to the equations of motion will doubtless be useful in the future.

Clearly the assumptions that we make are severe, and can only be justified by a combination of physical insight and experimentation. We now consider how to proceed experimentally. Experiments on arteries *in situ* are extremely difficult and prone to large errors; accurate experiments are possible only if we can make a physical model of the artery and have all the significant parameters under our control. If possible we should like the physical model to be large enough to facilitate accurate measurements, and we consider now how the system can be scaled in size.*

*SAQ 7*

Suppose that the equation

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)$$

models the real system, that $r$ and $t$ denote the radius and time, and that the density $\rho$ and kinematic viscosity $\nu\ (=\mu/\rho)$ are given. We wish to make a model larger than the real system and are free to scale the radius and time independently, i.e. if $\tilde{r}$ and $\tilde{t}$ denote the radius and time in the model, we are free to choose two independent parameters $\alpha$ and $\beta$ such that $r = \alpha\tilde{r}, \quad t = \beta\tilde{t}$.

Show that for the model to be similar to the real system, i.e. for the model to produce results which can be compared with the real situation, the kinematic viscosity of the fluid in the model must be given by $\tilde{\nu} = \beta\nu/\alpha^2$, and that if the pressure gradient is periodic, with period $T$, the model period is $T/\beta$ and the maxima and minima of the pressure gradient of the model are $\beta$ times those of the real system.

(Solution on p. 33.)

For the experiment with which we compare our results in Section 16.5, the important parameters are given in the following table; they correspond to the choice of $\alpha = 0.118$, $\beta = 0.13$.

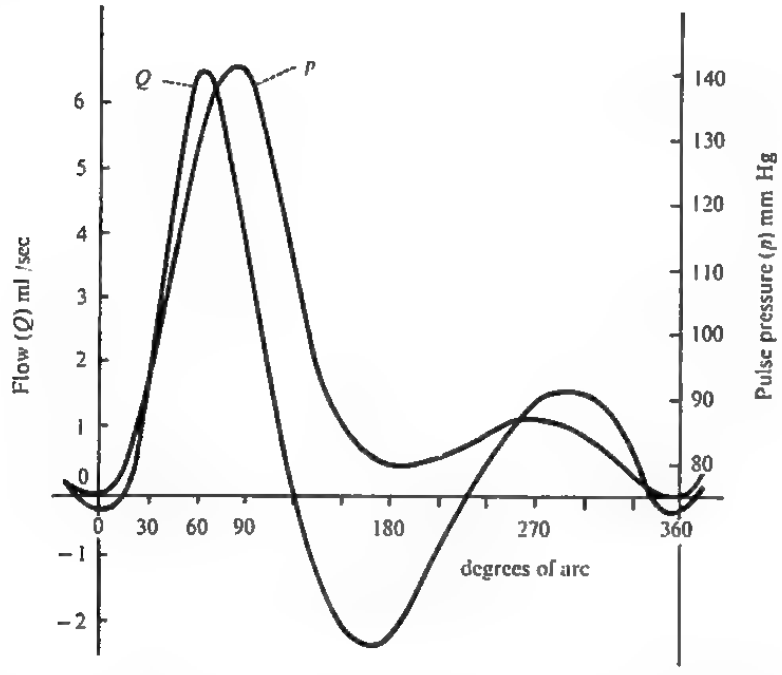
| | Artery | Model |
|---|---|---|
| radius | 0.3 cm | 2.54 cm |
| viscosity | 2.8 cp | 26.1 cp |
| period | 0.84 sec | 6.4 sec |
| pressure variation | 90–120 mm Hg | 11.7–15.6 mm Hg |

* Change of scale was discussed in *Unit 6, Fourier Series*

## 16.4 A TIME-DEPENDENT PRESSURE GRADIENT

### 16.4.1 Analysis into Fourier Components

The solution to the equation of motion (Equation (3) in Section 16.2.2) clearly depends upon the pressure gradient $\partial p/\partial x$, and in the case when $\partial p/\partial x$ is constant we have already solved the equation (SAQ 6). However, it is obvious that owing to the pumping action of the heart the pressure gradient in an artery is not constant. In this section we consider the form of the pressure gradient both in general and in the particular case of an artery of a dog, the latter in order that we can compare the theoretical results with observations.

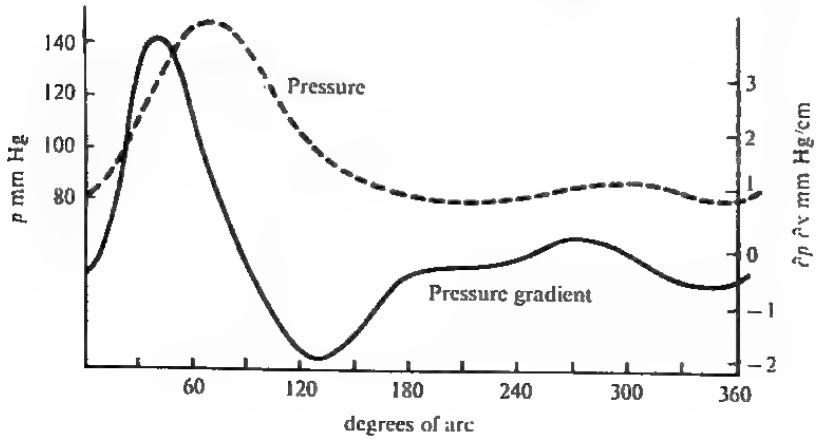



A flow velocity pulse and the arterial pressure pulse recorded simultaneously in an artery of a dog.

The abscissa representing time in this and all subsequent curves is marked with fractions of the cycle length in degrees of arc. The pulse frequency was 2.75 cycles/sec (period 0.36 sec) in this experiment. With pulse frequencies of this order one degree of arc represents 1 millisecond.

The figure above shows how the pressure $p$ and the rate of flow $Q$, at a point in an artery, vary with time through one period of the heart beat of a dog. However, it is not the pressure that we require, but the pressure gradient, which is determined, in practice, by recording the pressure at two points along the artery and subtracting the pressure of the downstream point from that at the upstream point at each time. A record of such a measurement is shown in the next figure.

Simultaneous recordings of a pressure pulse and the pressure gradient measured between two points 5 cm apart, in the femoral artery of a dog.



We shall suppose that the pressure gradient at $x = x_0$, say, has the same time-dependence at each heart-beat. Then the pressure gradient will be a periodic function of time and may be expressed as a Fourier series, so that we may write

$$\left(\frac{\partial p}{\partial x}\right)_{x=x_0} = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \tag{1}$$

where $c_n = \bar{c}_{-n}$, since the pressure is real, and $\omega = 2\pi/T$ where $T$ is the period. If we can decompose the pressure gradient curve in such a way, then, since the equation of motion is linear, each harmonic may be treated separately and the results added.

In all previous applications of Fourier series, the function being analysed was given analytically so that the components could be obtained by integration either analytically or numerically; whichever way we used, each component could be obtained to any required degree of accuracy. The present situation is different; the pressure gradient is given numerically by an experiment so that it is necessary to use a numerical technique for determining the components. However, there are experimental errors so that only the first few terms of the series can be found accurately; the remaining terms, being small, have the same size as the experimental errors. In order to compute the coefficients it is convenient to write Equation (1) in the form

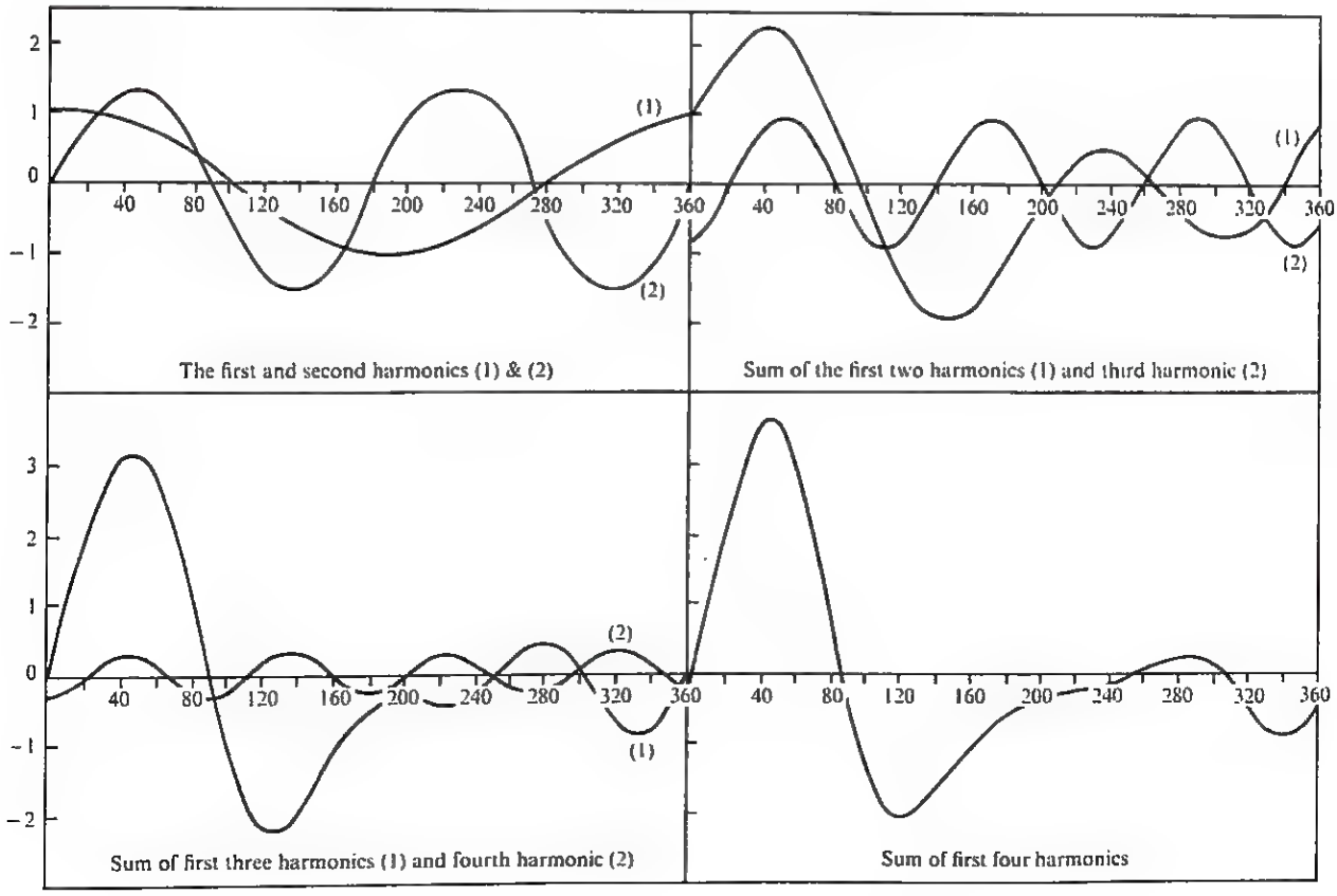
$$\frac{\partial p}{\partial x} = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t),$$

although (1) is the best form to use for solving the equation. The coefficients of the first four harmonics for the pressure gradient shown in the preceding figure are given below. (The $n$th **harmonic** is $a_n \cos n\omega t + b_n \sin n\omega t$.) The effect of summing these four harmonics is shown in the next figure.

*Fourier analysis of the pressure gradient (units mm Hg/cm)*

| $i$ | $a_i$ | $b_i$ |
|---|---|---|
| 1 | +1.024 | +0.240 |
| 2 | −0.126 | +1.346 |
| 3 | −0.819 | +0.477 |
| 4 | −0.305 | +0.002 |

$a_0$ = mean value = +0.335 mm Hg/cm



## 16.4.2 Oscillatory Flow Along a Cylinder

In Section 16.2.2 (SAQ 5) we derived the equation of motion for time-dependent flow along a cylinder,

$$\frac{\partial w}{\partial} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \qquad t \in (-\infty, \infty), \quad r \in (0, R), \tag{2}$$

where $\nu = \mu/\rho$ and $R$ is the radius of the cylinder. We now consider the special case where the pressure gradient is given by

$$-\frac{\partial p}{\partial x} = Ae^{i\omega t}, \tag{3}$$

where $A$ is a constant. When this solution has been obtained we can use the results of the previous section to determine the flow for a more realistic pressure gradient.

We expect the solution to be periodic in response to the forcing term (3) so that we may write

$$w(r, t) = W(r)e^{i\omega t};$$

when this is inserted into Equation (2) we obtain

$$\frac{d^2W}{dr^2} + \frac{1}{r}\frac{dW}{dr} - i\frac{\omega}{\nu}W = -\frac{A}{\nu\rho} \qquad 0 < r < R, \tag{4}$$

with the boundary conditions

$$W(R) = 0$$

and

$$W(r) \text{ is bounded as } r \rightsquigarrow 0.$$

*SAQ 8*

By making a suitable transformation, show that Equation (4) may be written in the form

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + u = \frac{A}{i\omega\rho},$$

and hence show that the solution to (4) is

$$W(r) = \frac{A}{i\omega\rho}\left[1 - \frac{J_0(i^{\frac{3}{2}}\alpha r/R)}{J_0(i^{\frac{3}{2}}\alpha)}\right], \tag{5}$$

where $\alpha = R(\omega/\nu)^{\frac{1}{2}}$.

HINT: See *W: page 179.*

(Solution on p. 33.)

*SAQ 9*

Using the result of SAQ 8, the full solution to (2) with the pressure gradient (3) and the boundary condition $w(R, t) = 0$ is

$$w(r, t) = \frac{A}{i\omega\rho}\left[1 - \frac{J_0(i^{\frac{3}{2}}\alpha r/R)}{J_0(i^{\frac{3}{2}}\alpha)}\right]e^{i\omega t}. \tag{6}$$

If $z$ is small $J_0(z)$ can be approximated by the first few terms of its series expansion (Equation (40.4) in *W: page 179*),

$$J_0(z) = 1 - \frac{z^2}{4} + O(z^4).$$

Using this result show that, in the appropriate limit, the solution (6) agrees with the steady state solution obtained in the solution to SAQ 6.

(Solution on p. 34.)

One of the quantities of interest is the total *flux* or rate of flow, $Q(\alpha, t)$, which is given by

$$Q(\alpha, t) = \int_0^R \int_0^{2\pi} w(r, t) r \, dr \, d\theta$$

$$= 2\pi R^2 e^{i\omega t} \int_0^1 y W(Ry) dy,$$

where we have substituted $y = r/R$. Note that the flux is given by a complex expression since we are working with a complex pressure gradient. The physical interpretation is that the real and imaginary parts of $Q$ give the fluxes corresponding to pressure gradients which are the real and imaginary parts of $\partial p/\partial x$, respectively.

*SAQ 10*

(a) Show that the rate of steady flow of a viscous fluid along a cylindrical pipe of radius $R$, and with a constant pressure gradient $-A$, is

$$Q_s = \frac{\pi A R^4}{8\mu}.$$

The suffix s is used to denote the fact that the flow is steady.

(b) Using the relation

$$\int_0^1 y J_0(\xi y) dy = \frac{1}{\xi} J_1(\xi),$$

which can be deduced from Equation (40.6) in *W: page 180*, show that

$$Q(\alpha, t) = \frac{\pi R^2 A}{i\omega\rho}\left[1 - \frac{2J_1(i^{\frac{3}{2}}\alpha)}{i^{\frac{3}{2}}\alpha J_0(i^{\frac{3}{2}}\alpha)}\right] e^{i\omega t}$$

$$= \frac{8Q_s}{i\alpha^2}\left[1 - \frac{2J_1(i^{\frac{3}{2}}\alpha)}{i^{\frac{3}{2}}\alpha J_0(i^{\frac{3}{2}}\alpha)}\right] e^{i\omega t}. \qquad (7)$$

(Solution on p. 34.)

In general the terms $J_n(i^{\frac{3}{2}}\alpha)$ cannot be computed from a simple formula and an involved numerical procedure must be used; in the appendix we give graphs for the real and imaginary parts of these functions in the cases $n = 0, 1$. However, when $\alpha \ll 1$ or $\alpha \gg 1$ there are closed forms which are good approximations to the function. For $\alpha \ll 1$ we may use the first few terms of the series expansion

$$J_n(i^{\frac{3}{2}}\alpha) = \frac{i^{3n/2}\alpha^n}{2^n n!}\left[1 + \frac{i\alpha^2}{4(n+1)} + O(\alpha^4)\right],$$

as in SAQ 9.

*SAQ 11*

Show that for small $\alpha$ (low frequencies)

$$Q(\alpha, t) \simeq Q_s e^{i\omega t},$$

i.e., when the period is long, the fluid has time to adjust to the slow changes in the pressure gradient.

(Solution on p. 35.)

If on the other hand $\alpha \gg 1$ we may use the asymptotic form of the Bessel function (see, for example, M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, 1965), p. 364),

$$J_n(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left[z - \left(\frac{n}{2} + \frac{1}{4}\right)\pi\right],$$

to give, after some manipulation,

$$J_n(i^{\frac{3}{2}}\alpha) \sim \left(\frac{1}{2\pi\alpha}\right)^{\frac{1}{2}} e^{\alpha 2^{-\frac{1}{2}}} \exp i\left[\frac{\alpha}{2^{\frac{1}{2}}} + \left(\frac{4n-1}{8}\right)\pi\right].$$

*SAQ 12*

Show that for $\alpha \gg 1$

$$Q(\alpha, t) \sim \frac{8Q_s}{i\alpha^2}\left(1 - \frac{2}{\alpha}e^{-i(\pi/4)}\right)e^{i\omega t}$$

$$\sim \frac{8Q_s}{\alpha^2}e^{i(\omega t - \pi/2)},$$

i.e. if the pressure gradient changes rapidly, the fluid motion is always out of phase and behind the pressure fluctuations.

(Solution on p. 35.)

Having determined $Q$ in the form of Equation (7) it is necessary to evaluate the expression. We need to know the typical values of $\alpha$, some of which are shown in the table below.

*Some values of $\alpha$ in different mammalian species*

| *Species* | *Pulse rate* ($min^{-1}$) | *Radius* (*cm*) | $\alpha = R(\omega/\nu)^{\frac{1}{2}}$ |
|---|---|---|---|
| | | Site – root of aorta | |
| mouse | 600–730 | 0.03 –0.04 | 1.19– 1.74 |
| rat | 360–520 | 0.045–0.095 | 1.38– 3.5 |
| rabbit | 205–220 | 0.17 | 3.92– 4.07 |
| cat | 180 | 0.2 | 4.4 |
| dog | 72–125 | 0.55 –0.6 | 8.27–10.68 |
| man | 55–72 | 1.08 –1.11 | 13.5 –16.7 |
| ox | 43 | 2.0 | 21.1 |
| elephant | 40–50 | 4.47 | 48–51 |
| | | Site – femoral artery | |
| rabbit | 210–360 | 0.06 | 1.4 – 1.8 |
| dog | 72–180 | 0.12 –0.15 | 1.65– 3.25 |
| man | 60–72 | 0.2 –0.25 | 2.5 – 3.5 |

The range of average body weights of these species is from 35 g (mouse) to 2.0–2.5 × $10^6$ g (elephant). The higher range of pulse rate allowed in the femoral artery of the rabbit and dog are based on observations in anaesthetized animals.

If $\alpha \simeq 1$, which it is in a large number of cases, there is no simple approximation to the Bessel function, and it is necessary to resort to tables.

Using Equation (7) and the Fourier decomposition of the pressure gradient we can find the rate of flow for any given periodic pressure variation. In the first instance we notice that the mean value of $Q(\alpha, t)$ is zero except when $\alpha = 0$ (in which case it is $Q_s$), so that none of the periodic components of the pressure gradient contributes to the overall flow. However, they do contribute to the energy dissipated by the system, as we shall now see.

## 16.4.3 Dissipation of Energy

We now consider the energy required to maintain the flow. The *energy per unit volume* is simply the kinetic energy $\frac{1}{2}\rho w^2$, so that the energy in a tube of length $l$ is

$$E(t) = \int_x^{x+l} \left\{ \int_0^R (\tfrac{1}{2}\rho w^2) 2\pi r \, dr \right\} dx.$$

The energy itself is of no interest; the interesting quantity is the *mean rate of change of the energy*, which measures the average rate at which the heart is doing work on this part of the system. Thus we first calculate the rate of change of energy

$$\frac{dE}{dt} = 2\pi\rho \int_x^{x+l} \int_0^R w \frac{\partial w}{\partial t} r \, dr \, dx.$$

*SAQ 13*

Using the equation of motion (Equation (2) in Section 16.4.2) and the equation above show that

$$\frac{dE}{dt} = \frac{dE^{(1)}}{dt} + \frac{dE^{(2)}}{dt}$$

where

$$\frac{dE^{(1)}}{dt} = -Q(\alpha, t)\big(p(x + l, t) - p(x, t)\big)$$

and

$$\frac{dE^{(2)}}{dt} = -2\pi l\mu \int_0^R \left(\frac{\partial w}{\partial r}\right)^2 r \, dr.$$

(Solution on p. 35.)

The first of these terms represents the rate at which work is being done to maintain the pressure difference; if the pressure is periodic in time with zero mean value the mean of this quantity is zero. The second term is the rate at which work is being done against the viscous forces; it is always negative (unless $w \equiv 0$).

In deriving the result in SAQ 13 it is implicitly assumed that the velocity $w$ is real; in our calculations $w$ is complex because we chose a complex pressure gradient in Equation (3). To make the velocity real we consider the pressure gradient

$$\frac{\partial p}{\partial x} = -Ae^{i\omega t} - \bar{A}e^{-i\omega t} = 2[\mathrm{Im}(A)\sin\omega t - \mathrm{Re}(A)\cos\omega t]$$

which is real. Since the equation of motion is linear, the velocity is, for this pressure gradient,

$$w(r, t) = W(r)e^{i\omega t} + \overline{W(r)}e^{-i\omega t},$$

which is also real, so that

$$\left(\frac{\partial w}{\partial r}\right)^2 = 2\left|\frac{dW}{dr}\right|^2 + \left(\frac{dW}{dr}\right)^2 e^{2i\omega t} + \left(\frac{d\overline{W}}{dr}\right)^2 e^{-2i\omega t} \qquad (8)$$

We consider now the mean value of $dE^{(2)}/dt$ over one cycle,

$$\left\langle \frac{dE^{(2)}}{dt} \right\rangle = -2\pi l\mu \int_0^R \left\langle \left| \frac{\partial w}{\partial r} \right|^2 \right\rangle r\, dr,$$

where $\langle\ \rangle$ denotes the time average defined by

$$\langle f(t) \rangle = \frac{1}{T}\int_0^T f(t)dt,$$

$T$ being the period. Since $\langle e^{in\omega t} \rangle = 0$, we find, using Equation (8), that

$$\left\langle \frac{dE^{(2)}}{dt} \right\rangle = -4\pi l\mu \int_0^R \left| \frac{dW}{dr} \right|^2 r\, dr.$$

If we substitute for $W$ using the result of SAQ 8 we find that

$$\left\langle \frac{dE^{(2)}}{dt} \right\rangle = -4\pi l\mu \left| \frac{A}{\omega\rho} \right|^2 \left| \frac{\xi}{J_0(\xi)} \right|^2 \int_0^1 yJ_1(\xi y)J_1(\bar{\xi}y)dy$$

where $\xi = i^{\frac{3}{2}}\alpha$. On using the result

$$\int_0^1 yJ_1(uy)J_1(vy)dy = \frac{1}{u^2 - v^2}[uJ_2(u)J_1(v) - vJ_1(u)J_2(v)],$$

which can be obtained, using Equations (40.5) and (40.6) in *W: pages 179 and 180*, after several lines of manipulation, we find that the mean energy dissipated per unit time per unit length is

$$\left\langle \frac{1}{l}\frac{dE^{(2)}}{dt} \right\rangle = -4\pi \frac{R^4}{\mu\alpha^4} \left| \frac{A\xi J_1(\xi)}{J_0(\xi)} \right|^2 \mathrm{Re}\left[ \frac{J_2(\xi)}{\xi J_1(\xi)} \right]. \tag{9}$$

*SAQ 14*

(a) By using the recurrence relation (which may be obtained as in SAQ 2 of *Unit 14*)

$$J_{n-1}(z) + J_{n+1}(z) = \frac{2n}{z} J_n(z)$$

with $z = \xi = i^{\frac{3}{2}}\alpha$, where $\alpha$ is real, show that Equation (9) reduces to

$$\left\langle \frac{1}{l}\frac{dE^{(2)}}{dt} \right\rangle = \frac{4\pi|A|^2R^4}{\mu\alpha^4} \mathrm{Re}\left[ \frac{\xi J_1(\xi)}{J_0(\xi)} \right].$$

(b) Deduce that in the limiting cases $\alpha \ll 1$ and $\alpha \gg 1$ we obtain

$$\left\langle \frac{1}{l}\frac{dE^{(2)}}{dt} \right\rangle \simeq -\frac{\pi|A|^2R^4}{4\mu} \qquad \alpha \ll 1,$$

$$\left\langle \frac{1}{l}\frac{dE^{(2)}}{dt} \right\rangle \simeq -\frac{4\pi|A|^2R^4}{\mu\alpha^3 2^{\frac{1}{2}}} \qquad \alpha \gg 1.$$

(Solution on p. 36.)

## 16.5 COMPARISON WITH EXPERIMENTS

### 16.5.1 Sinusoidal Flow

For our first comparison we consider the simple case where the pressure gradient is purely sinusoidal,

$$\frac{\partial p}{\partial x} = a \sin \omega t \qquad \text{with } a \text{ real,}$$

so that the velocity and flow rates are given by the imaginary parts of Equations (6) and (7) in Section 16.4, respectively. There are two reasons for making this comparison. The first reason is that, until the experiment quoted here had been done, there were few experiments measuring the properties of time-dependent viscous flow, and the second is that this experiment provided a good check on the apparatus which was subsequently to be used for more complicated pressure gradients.

To compare the velocity result with experiment we need to evaluate the Bessel functions contained therein; we shall not go into this detail here. The numerical values chosen for the experiment ought to be fairly close to the scaled values determined in Section 16.3; we choose the following.

$$a = 207.7 \text{ N m}^{-3} = 1.56 \text{ mm Hg/m}$$

$$T = 6.4 \text{ sec}$$

$$\omega = 0.98 \text{ rad/sec}$$

$$\mu = 26.0 \text{ cp}$$

$$R = 2.54 \text{ cm}$$

$$\alpha = 5.38$$

The experimental results and the numerical evaluation of the velocity from the theoretical solution are shown in the next figure. In general we see that the agreement is good; near the beginning of the cycle the experimental results are ahead of the theoretical ones, whilst the converse is the case towards the end of the cycle. This is probably due to experimental error.

Let us now consider the rate of dissipation of energy due to viscous forces. For the experiment considered above $\alpha = 5.38$, and for such large $\alpha$ each Bessel function is given by the first term in its asymptotic expansion to within about 5%. Thus, with reasonable accuracy, we may use the formula obtained in SAQ 14. Substituting the above numbers, we find that

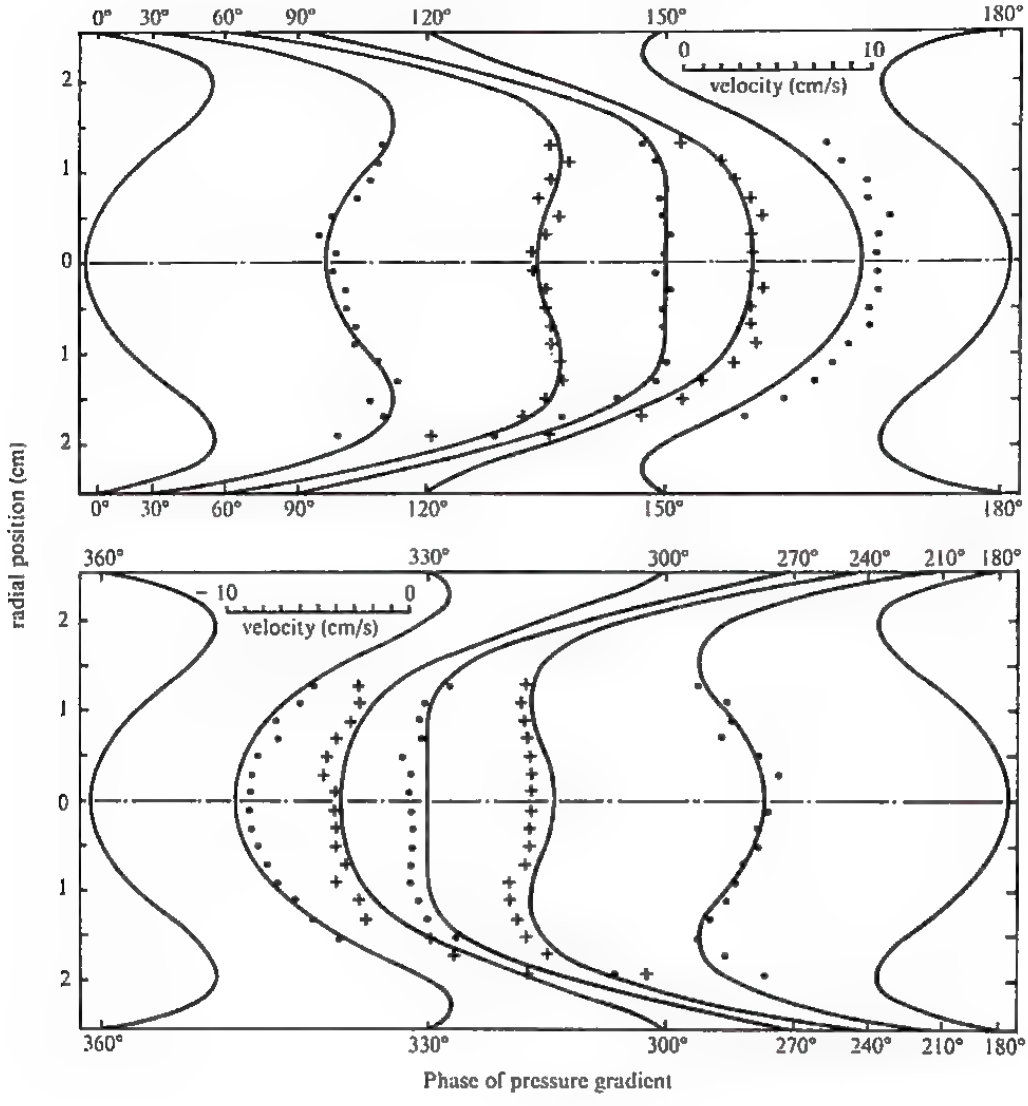
$$\left\langle \frac{1}{l}\frac{dE^{(2)}}{dt} \right\rangle \simeq -\frac{4\pi A^2 R^4}{\mu \alpha^3 2^{\frac{1}{2}}} \qquad \text{where } A = 0.5a$$

$$\simeq 10^{-2} \text{ watts m}^{-1}$$

$$\simeq 4 \times 10^{-6} \text{ horse power ft}^{-1},$$

which is very small. For the arterial system, using the data given in Section 16.3, and assuming a pressure gradient variation of 3mm Hg cm$^{-1}$ we get a figure of about $10^{-4}$ h.p. ft$^{-1}$; this is again negligible. These results suggest that there are other, better mechanisms for absorbing the energy; one would be the flow through small capillaries.

Comparison of theoretical and experimental velocity profiles for sinusoidal flow in a rigid pipe.




solid line – theoretical
+ • – experimental

## 16.5.2 Actual Observations

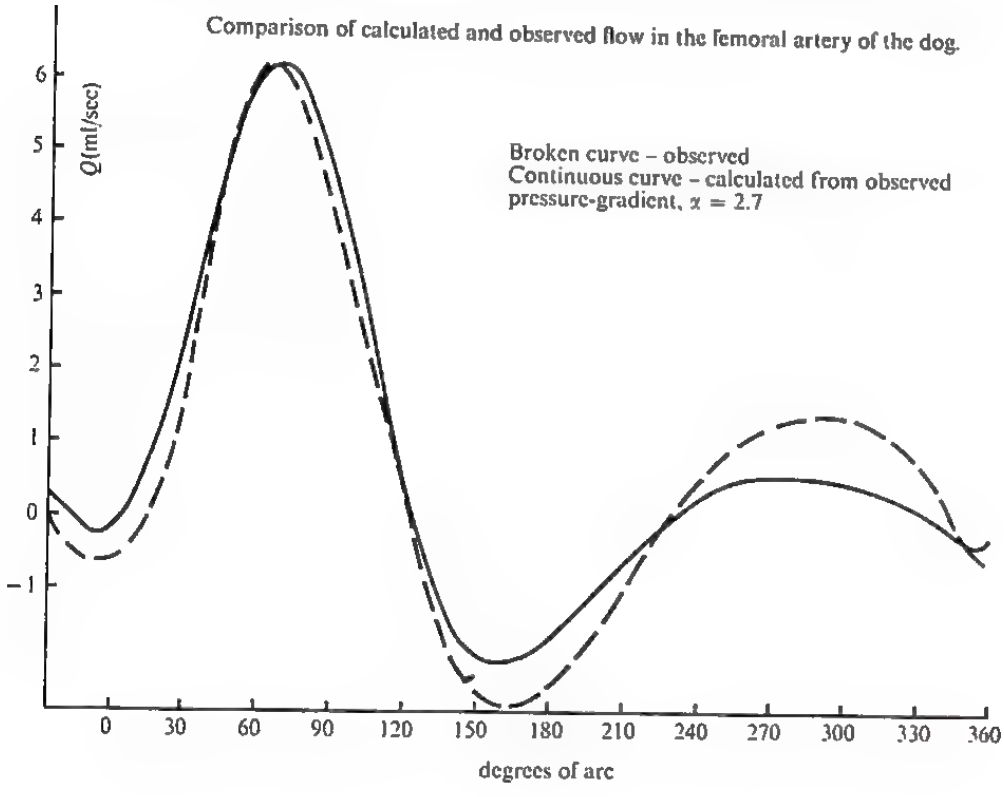
In order to compare our theory with experiment, we first measure the pressure gradient and perform a Fourier analysis on the result to obtain

$$-\left(\frac{\partial p}{\partial x}\right)_{x=x_0} = a_0 + \sum_{n=1}^{N} (a_n \cos n\omega t + b_n \sin n\omega t) \tag{1}$$

where $N$ is some number (about 4) determined by the accuracy of the experiment. In Section 16.4 we calculated the flow $Q(\alpha, t)$ for the complex pressure gradient $-Ae^{i\omega t}$. It follows by linearity that the rate of flow, with the pressure gradient given by Equation (1), is

$$Q(t) = a_0 Q_s + \sum_{n=1}^{N} \{a_n \operatorname{Re}[Q(\alpha\sqrt{n}, t)] + b_n \operatorname{Im}[Q(\alpha\sqrt{n}, t)]\}. \tag{2}$$

In the next figure we show a comparison of (2) with actual observations on a dog; in this case $\alpha = 2.7$ and our results are taken from Womersley (see Bibliography). Considering the simplicity of our model the agreement is remarkably good.



## 16.6 SUMMARY

An attempt has been made to indicate how some of the mathematics developed in this course is used in practice. The work described here is necessarily of limited validity; the extension that takes account of the elasticity of the wall is made in the paper by Womersley. However, as we saw in Section 16.3, there are many other inaccuracies inherent in this model.

It has been shown that mathematics on its own is of limited use in obtaining insight into a physical situation and that only by using a variety of techniques do we begin to understand a physical system; this simple fact is neglected in some applied mathematics texts.

The methods of attack in the two case studies, *Unit 7, Overhead Wires* and this unit, are typical of the approach towards solving physical problems by use of the applied mathematics developed in this course.

## 16.7 SOLUTIONS TO SELF-ASSESSMENT QUESTIONS

*Solution to SAQ 1*

In this case we consider the $y$-component of the equation of motion which is

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + (F + \text{viscous force}),$$

since $(\mathbf{v} \cdot \mathbf{grad})\mathbf{v} = \mathbf{0}$ when $\mathbf{v}(\mathbf{r}, t) = (0, v(z, t), 0)$. For this flow we have shown that the viscous forces are given by Equation (6) in Section 16.1, so that the full equation of motion becomes

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial z^2} + F.$$

*Solution to SAQ 2*

(a) The solution follows as in SAQ 1 by noting that the pressure is constant, that the external forces are zero, and that $\mathbf{v}(\mathbf{r}, t) = (0, v(x, t), 0)$. Thus we have

$$\frac{\partial v}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2 v}{\partial x^2} = \nu \frac{\partial^2 v}{\partial x^2}.$$

On the boundary $x = 0$, the fluid velocity is the same as that of the boundary, so that

$$v(0, t) = u(t).$$

(b) It is clear that, because of the oscillatory motion of the boundary, the fluid motion at any point will be oscillatory, and because the equation is linear the frequency of the oscillation will be that of the boundary. The problem is simplified by imposing the complex boundary condition $\tilde{u}(t) = Ae^{i\omega t}$ and taking the real part of the solution obtained with this boundary condition.

Since we expect an oscillatory solution we consider a solution of the form

$$\tilde{v}(x, t) = V(x)e^{i\omega t},$$

which gives, on substitution into the equation of motion,

$$\nu \frac{d^2 V}{dx^2} - i\omega V = 0.$$

The general solution to this is

$$V(x) = \alpha \exp\left[-\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}(1 + i)x\right] + \beta \exp\left[\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}(1 + i)x\right],$$

and, since $\tilde{v}(x, t)$ is bounded as $x \rightsquigarrow \infty$, we must have $\beta = 0$; then the boundary condition at $x = 0$ gives $\alpha = A$, and

$$\tilde{v}(x, t) = A \exp(-kx) \exp i(\omega t - kx)$$

where

$$k = \left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}.$$

The real part of this is

$$v(x, t) = Ae^{-kx} \cos(\omega t - kx),$$

which represents a wave travelling into the fluid with speed $\omega/k = (2\nu\omega)^{\frac{1}{2}}$ and with an exponentially decaying amplitude.

The force per unit area is given by Equation (3) of Section 16.1 as

$$\mu \left(\frac{\partial v}{\partial x}\right)_{x=0}.$$

But

$$\frac{\partial v}{\partial x} = kAe^{-kx}\{\sin(\omega t - kx) - \cos(\omega t - kx)\}$$
$$= -\sqrt{2}Ake^{-kx}\cos(\omega t - kx + \tfrac{1}{4}\pi),$$

so that

$$\mu\left(\frac{\partial v}{\partial x}\right)_{x=0} = -\sqrt{2}Ak\mu\cos(\omega t + \tfrac{1}{4}\pi).$$

This force is out of phase with the motion, i.e. the maximum amplitude of the force does not coincide with that of the motion.

*Solution to SAQ 3*

(a) (i) The motion is clearly independent of the time so that $\partial v/\partial t = 0$ and Equation (2) in Section 16.2.1 reduces to

$$v'' + \frac{v'}{r} - \frac{v}{r^2} = 0.$$

The general solution to this is

$$v(r) = Ar + \frac{B}{r},$$

and the boundary conditions $v(a) = 0$ and $v(b) = b\Omega$ give

$$Aa^2 + B = 0,$$
$$Ab^2 + B = \Omega b^2;$$

so that

$$A = \frac{\Omega b^2}{b^2 - a^2}, \quad B = \frac{-\Omega a^2 b^2}{b^2 - a^2},$$

and

$$v(r) = \frac{\Omega b^2}{b^2 - a^2}\left(\frac{r^2 - a^2}{r}\right).$$

(ii) In this case the solution is a little more awkward. The problem is similar to SAQ 2(b) and we use the same method to solve it. We impose a boundary condition at $r = b$ which is complex, $\tilde{v}(b, t) = Abe^{i\omega t}$, and extract the real part of the ensuing solution. As before we suppose that

$$\tilde{v}(r, t) = V(r)e^{i\omega t},$$

so that substitution into Equation (2) gives

$$\frac{i\omega\rho}{\mu}V = V'' + \frac{V'}{r} - \frac{V}{r^2},$$

or

$$z^2\frac{d^2V}{dz^2} + z\frac{dV}{dz} + (z^2 - 1)V = 0$$

where $z = \lambda r$, $\lambda^2 = -i\omega\rho/\mu$. This is Bessel's equation of order 1, which yields the general solution,

$$V(r) = \alpha J_1(\lambda r) + \beta Y_1(\lambda r).$$

The boundary conditions at $r = a, b$ then give

$$\alpha J_1(\lambda a) + \beta Y_1(\lambda a) = 0$$

and

$$\alpha J_1(\lambda b) + \beta Y_1(\lambda b) = Ab,$$

so that

$$\alpha = \frac{AbY_1(\lambda a)}{\gamma}, \quad \beta = -\frac{AbJ_1(\lambda a)}{\gamma}$$

where

$$\gamma = J_1(\lambda b)Y_1(\lambda a) - J_1(\lambda a)Y_1(\lambda b).$$

The solution is thus

$$\tilde{v}(r, t) = \frac{Ab}{\gamma}\{J_1(\lambda r)Y_1(\lambda a) - Y_1(\lambda r)J_1(\lambda a)\}e^{i\omega t}.$$

The solution for our problem is then the real part of this, which may be obtained numerically.

(b) The moment on the inner cylinder is (per unit length)

$$2\pi a^2 \mu \left(\frac{\partial v}{\partial r}\right)_{r=a} = \frac{4\pi\mu\Omega b^2 a^2}{b^2 - a^2}.$$

(c) The kinetic energy of an elementary ring of fluid of width $\Delta r$ is

$$\tfrac{1}{2} \text{ mass} \times (\text{velocity})^2 = \tfrac{1}{2}(2\pi r \Delta r \rho) + v^2$$
$$= \pi\rho r v^2 \Delta r.$$

The total kinetic energy is thus

$$E = \pi\rho \int_a^b r v^2 \, dr.$$

The rate of change is

$$\frac{dE}{dt} = \pi\rho \frac{d}{dt} \int_a^b r v^2 \, dr$$
$$= \pi\rho \int_a^b r \frac{\partial(v^2)}{\partial t} dr$$
$$= 2\pi\rho \int_a^b r v \frac{\partial v}{\partial t} dr.$$

(d) Substituting for $\partial v/\partial t$ from the equation of motion we obtain

$$\frac{dE}{dt} = 2\pi\mu \int_a^b \left( r v \frac{\partial^2 v}{\partial r^2} + v \frac{\partial v}{\partial r} - \frac{v^2}{r} \right) dr.$$

We integrate (by parts) the first term in the integral and obtain

$$\frac{dE}{dt} = 2\pi\mu \left\{ \left[ r v \frac{\partial v}{\partial r} \right]_a^b - \int_a^b \frac{\partial v}{\partial r} \left( v + r \frac{\partial v}{\partial r} \right) dr + \int_a^b \left( v \frac{\partial v}{\partial r} - \frac{v^2}{r} \right) dr \right\}$$
$$= 2\pi\mu \left[ r v \frac{\partial v}{\partial r} \right]_a^b - 2\pi\mu \int_a^b \left[ r \left( \frac{\partial v}{\partial r} \right)^2 + \frac{v^2}{r} \right] dr.$$

We consider the first term. The force per unit area of the boundary is $\mu \, \partial v/\partial r$, so that the rate of working of the boundary is $\mu(v \, \partial v/\partial r)_{r=\theta}$ per unit area, where $\theta = a$ or $b$. Thus the rate of working on a cylinder of length $l$ is $2\pi l \mu r v \, \partial v/\partial r$. This term thus represents the energy given to the fluid by a moving boundary. If the boundaries are stationary, $v(a, t) = v(b, t) = 0$ and no energy is lost or gained.

The second term is simply the energy lost owing to viscous forces in the liquid; the integrand is always positive and so this term is always negative. Thus the energy is always decreasing, and the fluid will ultimately come to rest; the energy is not annihilated but is converted into random motion, which means that the fluid gets heated.

*Solution to SAQ 4*

The viscous force is obtained from the formula

$$\text{force} = \mu \times \text{area} \times \text{velocity gradient}$$

$$= \mu 2\pi\tilde{r}l\left(\frac{\partial w}{\partial r}\right)_{r=\tilde{r}}$$

at a radius $\tilde{r}$. On the inner wall $\tilde{r} = r - \Delta r$ and on the outer wall $\tilde{r} = r + \Delta r$, whence the result quoted. The only other force acting on this element of fluid is that due to the pressure at either end, i.e.

$$\text{pressure} \times \text{area},$$

where the area is given by $2\pi r \times 2\Delta r$. Thus the total force in the positive $x$-direction is

$$-4\pi r\Delta r[p(x + l) - p(x)] + 2\pi\mu l\left[r\frac{\partial w}{\partial r}\right]_{r-\Delta r}^{r+\Delta r}.$$

*Solution to SAQ 5*

Newton's Second Law of Motion is

$$\text{mass} \times \text{acceleration} = \text{force},$$

i.e.

$$4\pi rl\rho\Delta r\frac{\partial w}{\partial t} = -4\pi r\Delta r\{p(x + l) - p(x)\} + 2\pi\mu l\left[r\frac{\partial w}{\partial r}\right]_{r-\Delta r}^{r+\Delta r}.$$

First we divide by $4\pi rl\rho\Delta r$ and let $l \rightsquigarrow 0$; then, since

$$\lim_{l\rightsquigarrow 0}\left\{\frac{p(x + l) - p(x)}{l}\right\} = \frac{\partial p}{\partial x},$$

we obtain

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{2r\rho\Delta r}\left[r\frac{\partial w}{\partial r}\right]_{r-\Delta r}^{r+\Delta r}.$$

Now proceeding to the limit as $\Delta r \rightsquigarrow 0$ and noting that

$$\lim_{\Delta r\rightsquigarrow 0}\left[\frac{\left(r\frac{\partial w}{\partial r}\right)_{r+\Delta r} - \left(r\frac{\partial w}{\partial r}\right)_{r-\Delta r}}{2\Delta r}\right] = \frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right),$$

we obtain

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right).$$

*Solution to SAQ 6*

For a steady flow $w$ is independent of the time; it is given that $\partial p/\partial x = -\Delta p$, so that we need to solve

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} = -\frac{\Delta p}{\mu} = -k, \text{ say}$$

or

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) = -kr.$$

The solution is simply,

$$w(r) = -\frac{kr^2}{4} + c\ln r + d,$$

where $c$ and $d$ are arbitrary constants. Now $w$ must be finite at the origin, so that $c = 0$; also $w(R) = 0$ so that $d = \frac{1}{4}kR^2$. Hence

$$w(r) = \frac{k}{4}(R^2 - r^2) = \frac{\Delta p}{4\mu}(R^2 - r^2).$$

Note that the equation for steady flow is just the equation

$$\nabla^2 w = -k$$

quoted in Section 3.2.2 of *Unit 3, Elliptic and Parabolic Equations*, with circular symmetry.

*Solution to SAQ 7*

The equation of motion is

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)$$

where $\nu = \mu/\rho$. On putting $r = \alpha\tilde{r}$, $t = \beta\tilde{t}$, it is easy to see that, if $\tilde{w}(\tilde{r}, \tilde{t}) = w(\alpha\tilde{r}, \beta\tilde{t})$,

$$\frac{\partial \tilde{w}}{\partial \tilde{t}} = -\frac{1}{\rho}\left(\beta\frac{\partial p}{\partial x}\right) + \tilde{\nu}\left(\frac{\partial^2 \tilde{w}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{w}}{\partial \tilde{r}}\right),$$

with

$$\tilde{\nu} = \frac{\beta\nu}{\alpha^2},$$

is the equation describing the fluid flow of the model. The latter equation describes flow with kinematic viscosity given by

$$\tilde{\nu} = \frac{\beta\nu}{\alpha^2},$$

and pressure gradient $\beta\,\partial p/\partial x$. Thus if the pressure gradient is oscillatory with period $T$ in the artery it must have period $T/\beta$ in the model, and its amplitude must be a factor $\beta$ times the arterial amplitude to yield the same velocity profile.

*Solution to SAQ 8*

Consider the substitution $z = (-i\omega/\nu)^{\frac{1}{2}}r$ as in *Unit 14* so that Equation (4) becomes

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + u = \frac{A}{i\omega\rho},$$

where $W(r) = u(\sqrt{-i\omega/\nu}\, r)$. The general solution to this, which is bounded as $z \rightsquigarrow 0$, is

$$u(z) = \frac{A}{i\omega\rho} + cJ_0(z),$$

so that

$$W(r) = \frac{A}{i\omega\rho} + cJ_0\left(i^{\frac{1}{2}}(\omega/\nu)^{\frac{1}{2}}r\right) \qquad \text{since } (-i)^{\frac{1}{2}} = (i^3)^{\frac{1}{2}} = i^{\frac{1}{2}}$$

$$= \frac{A}{i\omega\rho} + cJ_0(i^{\frac{1}{2}}\alpha r/R)$$

where in this last equation we have set

$$\alpha = R(\omega/\nu)^{\frac{1}{2}}.$$

The boundary condition, $W(R) = 0$, then gives

$$W(r) = \frac{A}{i\omega\rho}\left[1 - \frac{J_0(i^{\frac{1}{2}}\alpha r/R)}{J_0(i^{\frac{1}{2}}\alpha)}\right].$$

*Solution to SAQ 9*

Using the expansion we have

$$W(r) \simeq \frac{A}{i\omega\rho}\left[1 - \frac{\left(1 + \dfrac{i\alpha^2 r^2}{4R^2}\right)}{\left(1 + \dfrac{i\alpha^2}{4}\right)}\right]$$

$$\simeq \frac{A}{i\omega\rho}\left[1 - \left(1 + \frac{i\alpha^2 r^2}{4R^2}\right)\left(1 - \frac{i\alpha^2}{4}\right)\right]$$

where the last expression is obtained by using the binomial expansion and ignoring terms of order $\alpha^4$. Expanding the brackets and again ignoring all terms of order $\alpha^4$ we obtain

$$W(r) \simeq \frac{A\alpha^2}{4\omega\rho}\left(1 - \frac{r^2}{R^2}\right),$$

so that

$$w(r, t) \simeq \frac{A}{4\mu}(R^2 - r^2)e^{i\omega t}.$$

Now one way in which $\alpha$ can be small is if $\omega \ll 1$; thus, if the period of the oscillations is large, the fluid motion is approximately in phase with the pressure changes, i.e. the fluid has time to adapt to the changes. This would be expected on common-sense grounds.

In the limit as $\omega \rightsquigarrow 0$ we have the steady state problem, and our solution agrees with that obtained in SAQ 6, with $A = \Delta p$.

*Solution to SAQ 10*

(a) For steady flow we use the result of SAQ 6, with $A = \Delta p$,

$$w(r) = \frac{A}{4\mu}(R^2 - r^2).$$

The rate of flow is now

$$Q_s = 2\pi \int_0^R w(r) r \, dr$$

$$= \frac{\pi A}{2\mu} \int_0^R (R^2 r - r^3) \, dr$$

$$= \frac{\pi A R^4}{8\mu}.$$

This is of course the result which you obtained in SAQ 14 of *Unit 3*.

(b) The total flow is

$$Q(\alpha, t) = \frac{2\pi R^2 A}{i\omega\rho} e^{i\omega t} \int_0^1 \left[y - \frac{y J_0(i^{\frac{3}{2}}\alpha y)}{J_0(i^{\frac{3}{2}}\alpha)}\right] dy$$

$$= \frac{\pi R^2 A}{i\omega\rho}\left[1 - \frac{2J_1(i^{\frac{3}{2}}\alpha)}{i^{\frac{3}{2}}\alpha J_0(i^{\frac{3}{2}}\alpha)}\right] e^{i\omega t}$$

on using the given relation. Since, from (a),

$$Q_s = \frac{\pi A R^4}{8\mu} = \frac{\pi R^2 A}{i\omega\rho} \cdot \frac{i\alpha^2}{8}$$

the result follows.

*Solution to SAQ 11*

Consider the expression

$$1 - \frac{2J_1(i^{\frac{3}{2}}\alpha)}{i^{\frac{3}{2}}\alpha J_0(i^{\frac{3}{2}}\alpha)} = 1 - \frac{1 + \dfrac{i\alpha^2}{8}}{1 + \dfrac{i\alpha^2}{4}} + O(\alpha^4)$$

which we obtain on using the small $\alpha$ expansion. Using the Binomial Theorem this is equal to

$$1 - \left(1 + \frac{i\alpha^2}{8}\right)\left(1 - \frac{i\alpha^2}{4}\right) + O(\alpha^4) = \frac{i\alpha^2}{8} + O(\alpha^4),$$

and the result follows using the formula obtained in SAQ 10(b).

Alternatively, the result may be obtained directly by using the result of SAQ 9.

*Solution to SAQ 12*

Using the asymptotic form quoted we find that for $\alpha \gg 1$

$$\frac{J_1(i^{\frac{3}{2}}\alpha)}{J_0(i^{\frac{3}{2}}\alpha)} \sim \frac{\exp i\left(\dfrac{\alpha}{2^{\frac{1}{2}}} + \dfrac{3\pi}{8}\right)}{\exp i\left(\dfrac{\alpha}{2^{\frac{1}{2}}} - \dfrac{\pi}{8}\right)} = e^{i\pi/2},$$

so that, from Equation (7) in Section 16.4.2 we have

$$Q(\alpha, t) \sim \frac{8Q_s}{i\alpha^2}\left[1 - \frac{2e^{i\pi/2}}{\alpha e^{3i\pi/4}}\right]e^{i\omega t} = \frac{8Q_s}{i\alpha^2}e^{i\omega t}\left(1 - \frac{2}{\alpha}e^{-i\pi/4}\right),$$

i.e.

$$Q(\alpha, t) \sim \frac{8Q_s}{\alpha^2}e^{i(\omega t - \pi/2)} \qquad \text{since } i = e^{i\pi/2}.$$

*Solution to SAQ 13*

Substituting the equation of motion into the expression for $dE/dt$ we can separate out two terms,

$$\begin{aligned}\frac{dE^{(1)}}{dt} &= -2\pi\int_x^{x+l}\int_0^R rw(r)\left(\frac{\partial p}{\partial x}\right)dr\,dx \\ &= -2\pi\int_0^R w(r)r\,dr\int_x^{x+l}\frac{\partial p}{\partial x}dx \\ &= -Q(\alpha, t)\big(p(x + l, t) - p(x, t)\big)\end{aligned}$$

and

$$\begin{aligned}\frac{dE^{(2)}}{dt} &= 2\pi\mu\int_x^{x+l}\int_0^R w(r)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)r\,dr\,dx \\ &= 2\pi\mu l\int_0^R w(r)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)r\,dr\end{aligned}$$

since $w(r)$ is independent of $x$. Integrating the first term of this last integral by parts we have

$$\begin{aligned}\frac{dE^{(2)}}{dt} &= 2\pi\mu l\left\{\left[rw\frac{\partial w}{\partial r}\right]_0^R - \int_0^R\left[w\frac{\partial w}{\partial r} + r\left(\frac{\partial w}{\partial r}\right)^2 - w\frac{\partial w}{\partial r}\right]dr\right\} \\ &= -2\pi\mu l\int_0^R r\left(\frac{\partial w}{\partial r}\right)^2 dr\end{aligned}$$

since $w(R, t) = 0$.

*Solution to SAQ 14*

(a) Putting $n = 1$ in the recurrence relation, we have

$$J_2(z) = \frac{2}{z}J_1(z) - J_0(z)$$

so that

$$\frac{J_2(\xi)}{\xi J_1(\xi)} = \frac{2}{\xi^2} - \frac{J_0(\xi)}{\xi J_1(\xi)},$$

and since $\xi = i^{\frac{3}{4}}\alpha$, $\xi^2 = -i\alpha^2$ is purely imaginary so that

$$\mathrm{Re}\left(\frac{J_2(\xi)}{\xi J_1(\xi)}\right) = -\mathrm{Re}\left(\frac{J_0(\xi)}{\xi J_1(\xi)}\right) = -\mathrm{Re}\left(\frac{1}{\Phi}\right), \text{ say.}$$

But

$$\mathrm{Re}\left(\frac{1}{\Phi}\right) = \mathrm{Re}\left(\frac{\Phi}{|\Phi|^2}\right) = \frac{1}{|\Phi|^2}\mathrm{Re}(\Phi).$$

Thus

$$\left|\frac{\xi J_1(\xi)}{J_0(\xi)}\right|^2 \mathrm{Re}\left(\frac{J_2(\xi)}{\xi J_1(\xi)}\right) = -\mathrm{Re}\left(\frac{\xi J_1(\xi)}{J_0(\xi)}\right),$$

and the result follows.

(b) If $\alpha \ll 1$ we have

$$\frac{\xi J_1(\xi)}{J_0(\xi)} \simeq \frac{\xi[\frac{1}{2}\xi(1 - \frac{1}{8}\xi^2)]}{1 - \frac{1}{4}\xi^2} \quad \text{using the first two terms of the series expansion}$$

$$\simeq -\tfrac{1}{2}i\alpha^2(1 + \tfrac{1}{8}i\alpha^2)(1 - \tfrac{1}{4}i\alpha^2)$$

$$\simeq -\tfrac{1}{2}i\alpha^2 - \tfrac{1}{16}\alpha^4.$$

Thus

$$\left\langle \frac{1}{l}\frac{dE^{(2)}}{dt} \right\rangle \simeq \frac{4\pi|A|^2R^4}{\mu\alpha^4} \cdot \left(-\frac{\alpha^4}{16}\right) = -\frac{\pi|A|^2R^4}{4\mu}.$$

If $\alpha \gg 1$ we have, as in the solution to SAQ 12,

$$\mathrm{Re}\left[\frac{\xi J_1(\xi)}{J_0(\xi)}\right] \simeq \mathrm{Re}(\xi e^{i\pi/2}) = \mathrm{Re}(-i^{\frac{1}{4}}\alpha) = -\alpha\,\mathrm{Re}\left(\frac{1+i}{2^{\frac{1}{4}}}\right) = -\frac{\alpha}{2^{\frac{1}{4}}}.$$

Thus

$$\left\langle \frac{1}{l}\frac{dE^{(2)}}{dt} \right\rangle \simeq -\frac{4\pi|A|^2R^4}{\mu\alpha^4} \cdot \frac{\alpha}{2^{\frac{1}{4}}} = -\frac{4\pi|A|^2R^4}{\mu\pi^3 2^{\frac{1}{4}}}.$$

# 16.8 APPENDIX

## Bessel Functions

Here we try to give you an idea of how the Bessel functions $J_0(i^{\frac{3}{2}}x)$ and $J_1(i^{\frac{3}{2}}x)$ behave. These functions have complex values and we examine their real and imaginary parts separately.

In the first two figures, we have plotted the real and imaginary parts of the three functions

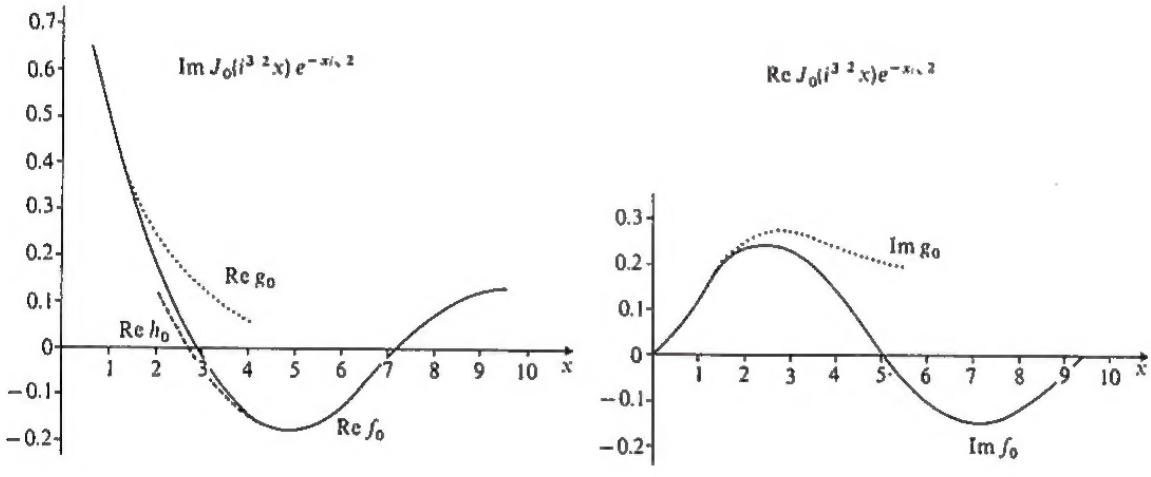
$$f_0(x) = J_0(i^{\frac{3}{2}}x)e^{-x/\sqrt{2}},$$

$$g_0(x) = \left(1 + \frac{ix^2}{4}\right) e^{-x/\sqrt{2}}$$

which approximates $f_0$ by taking the first two terms in the expansion of $J_0$, and

$$h_0(x) = \left(\frac{1}{2\pi x}\right)^{\frac{1}{2}} \exp i\left(\frac{x}{\sqrt{2}} - \frac{\pi}{8}\right).$$

obtained using the asymptotic form preceding SAQ 12.




We see that the power series expansion is good up to about $x = 2$, after which it becomes progressively worse. For $x \geqslant 2$ the asymptotic form for the imaginary part is indistinguishable from the correct values, and for the real part the same is true for $x \geqslant 4$. This is shown quite clearly in the following table.

| $x$ | Re $f_0(x)$ | Re $h_0(x)$ | Im $f_0(x)$ | Im $h_0(x)$ |
|---|---|---|---|---|
| 1.0 | 0.485 | 0.379 | 0.123 | 0.123 |
| 2.0 | 0.183 | 0.147 | 0.236 | 0.241 |
| 3.0 | −0.0265 | −0.0362 | 0.232 | 0.227 |
| 4.0 | −0.152 | −0.152 | 0.136 | 0.129 |
| 5.0 | −0.182 | −0.178 | 0.0034 | −0.0002 |
| 6.0 | −0.127 | −0.124 | −0.105 | −0.106 |

Notice that there is a region around $x = 3$ where both $g_0$ and $h_0$ are poor approximations. It is typical of these types of approximations that there is a region where neither works very well.

In the next pair of figures we show

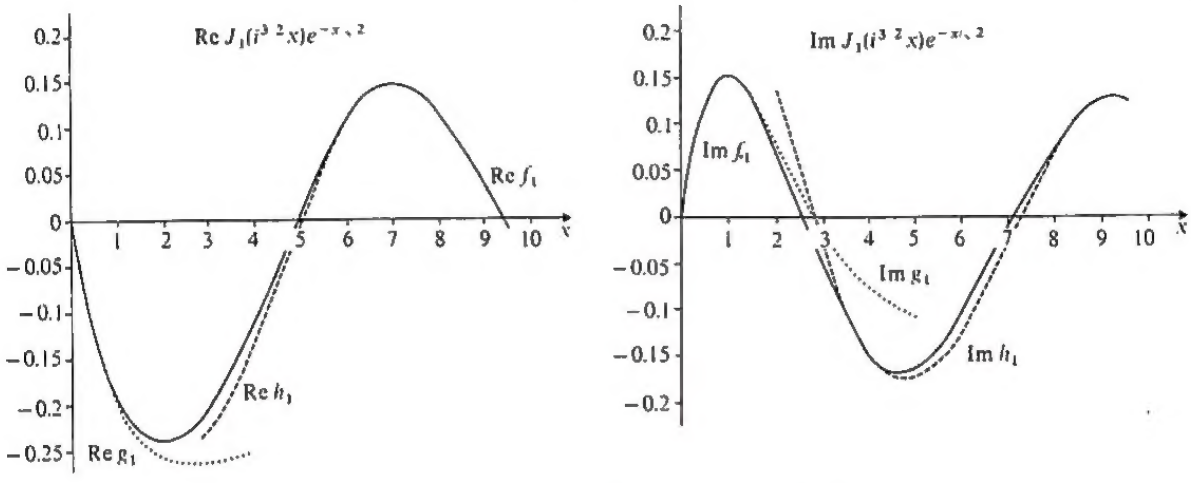
$$f_1(x) = J_1(i^{\frac{3}{2}}x)e^{-x/\sqrt{2}}$$

together with the form derived from the series expansion

$$g_1(x) = \frac{x}{2\sqrt{2}}\left[-1 - \frac{x^2}{8} + i\left(1 - \frac{x^2}{8}\right)\right]e^{-x/\sqrt{2}}$$

and the asymptotic form

$$h_1(x) = \left(\frac{1}{2\pi x}\right)^{\frac{1}{2}} \exp i\left(\frac{x}{\sqrt{2}} + \frac{3\pi}{8}\right).$$




The comments for these figures are much the same as for the previous pair, but notice that the asymptotic expansion is not very accurate until $x \simeq 5$ for the real part and $x \simeq 7$ for the imaginary part; this is shown clearly in the table below.

| $x$ | Re $f_1(x)$ | Re $h_1(x)$ | Im $f_1(x)$ | Im $h_1(x)$ |
|---|---|---|---|---|
| 1.0 | −0.195 | −0.123 | 0.152 | 0.379 |
| 2.0 | −0.242 | −0.241 | 0.0739 | 0.147 |
| 3.0 | −0.208 | −0.227 | −0.0584 | −0.0362 |
| 4.0 | −0.111 | −0.129 | −0.152 | −0.152 |
| 5.0 | 0.0105 | 0.0002 | −0.169 | −0.178 |
| 6.0 | 0.107 | 0.106 | −0.113 | −0.124 |
| 7.0 | 0.144 | 0.149 | −0.0164 | −0.0233 |
| 8.0 | 0.114 | 0.120 | 0.0757 | 0.0739 |
| 9.0 | 0.0357 | 0.0408 | 0.124 | 0.127 |

## PARTIAL DIFFERENTIAL EQUATIONS OF APPLIED MATHEMATICS

| | | |
|---|---|---|
| 1 | *W* | The Wave Equation |
| 2 | *W* | Classification and Characteristics |
| 3 | *W* | Elliptic and Parabolic Equations |
| 4 | | NO TEXT |
| 5 | *S* | Finite-Difference Methods I: Initial Value Problems |
| 6 | *W* | Fourier Series |
| 7 | *N* | Motion of Overhead Electric Train Wires |
| 8 | *S* | Finite-Difference Methods II: Stability |
| 9 | *W* | Green's Functions I: Ordinary Differential Equations |
| 10 | *W* | Green's Functions II: Partial Differential Equations |
| 11 | *S* | Finite-Difference Methods III: Boundary Value Problems |
| 12 | | NO TEXT |
| 13 | *W* | Sturm-Liouville Theory |
| 14 | *W* | Bessel Functions |
| 15 | *N* | Finite-Difference Methods IV: Parabolic Equations |
| 16 | *N* | Blood Flow in Arteries |

*The letter after the unit number indicates the relevant set book: N indicates a unit not based on either book.*

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